Finite groups of arbitrary deficiency

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Deficiency

A natural question to ask about a finitely presented group is how complicated its presentations must be, for instance in terms of the number of generators and relations. Deficiency is a way to measure how concise a presentation is.

Definition

The deficiency of a presentation is defined as the number of generators minus the number of relations. The *deficiency* $\operatorname{def}(G)$ of a group G is the maximum deficiency of its presentations.

A finite group has non-positive deficiency, since a group with positive deficiency has infinite abelianization.

Example

The surface group $\Sigma_g = \pi_1 S_g$ has deficiency $2g-1$:

$$\langle\, a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \,\rangle .$$

The free abelian group $\mathbb{Z}^n = \pi_1 \mathbb{T}^n$ has deficiency $n - \binom{n}{2}$:

$$\langle\, a_1, \ldots, a_n \mid [a_i, a_j] \text{ for } i < j \,\rangle .$$

How do we know that these natural presentations are the best possible, that is, that these groups do not admit presentations of larger deficiency? We can use the following classical bound:

$$\operatorname{def}(G) \leq \operatorname{rk}(H_1(G, \mathbb{Z})) - d(H_2(G, \mathbb{Z})).$$

Here, rk denotes torsion-free rank, and d denotes the minimal size of a generating set. A group (or presentation) that attains the bound is called efficient. Even finite groups can be very far from efficient.

Computing the group homology via classifying spaces, since

$$H_1(S_g, \mathbb{Z}) = \mathbb{Z}^{2g} \qquad H_2(S_g, \mathbb{Z}) = \mathbb{Z}$$
$$H_1(\mathbb{T}^n, \mathbb{Z}) = \mathbb{Z}^n \qquad H_2(\mathbb{T}^n, \mathbb{Z}) = \mathbb{Z}^{\binom{n}{2}}$$

we see that the presentations in the Example are efficient, and so the groups have the deficiency we claimed.

Kähler groups

A Kähler group is the fundamental group of a compact Kähler manifold. Serre proposed the problem of determining which finitely presented groups are Kähler groups in the 1950's, and this has been an active topic of research ever since. Basic examples include $\mathbb{Z}^{2g}$ (the fundamental group of a complex torus), surface groups, and all finite groups (by a theorem of Serre).

Theorem (Kotschick [K12])

A Kähler group cannot have deficiency an even positive integer.

Surface groups give examples of deficiency any odd positive integer. Kotschick moreover gave constructions of Kähler groups for all negative deficiencies, *except* for gaps at -5 and -7.

Theorem [G17]

For every integer $n < 0$, there exists a finite group G_n of deficiency n.

This completes Kotschick's classification of deficiencies of Kähler groups.

Corollary

Let n be an integer. Then there exists a Kähler group of deficiency n if and only if n is odd or non-positive.

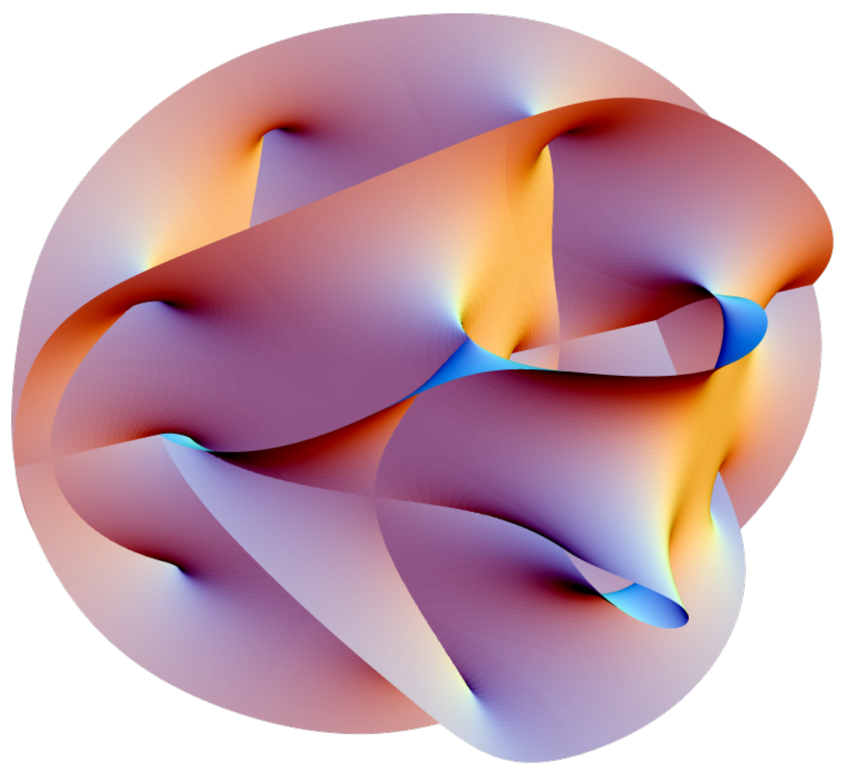

Figure: A cross-section of a Kähler manifold

source: http://en.wikipedia.org/wiki/File:Calabi-Yau.png

p-groups

In fact, one can arrange for G_n of the Theorem to be a p-group for any prime p, and there is an explicit description of G_n in terms of the following basic building blocks.

Fix a prime p, which we assume to be odd for simplicity.

$$A := \langle\, a \mid a^p \,\rangle$$
$$B := \langle\, a, b \mid a^p,\ b^p,\ [[a,b],a],\ [[a,b],b] \,\rangle$$
$$C := \langle\, a, b \mid a^{p^2},\ b^{p^2},\ [[a,b],a],\ [[a,b],b],\ [a,b]^p \,\rangle$$

So $A \cong \mathbb{Z}/p$ and B is the mod-p Heisenberg group. All three are finite p-groups, since they are nilpotent and generated by p-torsion elements.

We claim that the groups A, B, and C are efficient, with deficiency attained by their defining presentations. For A this is immediate: a finite group cannot have positive deficiency, so $\operatorname{def}(A) = 0$. It is well-known that $H_2(B, \mathbb{Z}) \cong \mathbb{Z}/p \times \mathbb{Z}/p$, so the above presentation is efficient, attaining $\operatorname{def}(B) = -2$.

Proposition

$$H_2(C, \mathbb{Z}) \cong \mathbb{Z}/p \times \mathbb{Z}/p \times \mathbb{Z}/p$$

It is more convenient to calculate $H^2(C, \mathbb{F}_p) \cong \mathbb{F}_p^5$ (from which the Proposition follows by the Universal Coefficient Theorem together with an easy upper bound on $H_2(C, \mathbb{Z})$). We do this via the Lyndon–Hochschild–Serre spectral sequence for $H^*(C, \mathbb{F}_p)$ associated to the (central) short exact sequence

$$1 \to \mathbb{Z}/p \to C \to \mathbb{Z}/p^2 \times \mathbb{Z}/p^2 \to 1.$$

The differentials d_2 and d_3 have a known description in this setting. With the extension class $\alpha \in H^2(\mathbb{Z}/p^2 \times \mathbb{Z}/p^2, \mathbb{Z}/p)$ that corresponds to C, we can use the ring structure on cohomology to determine the relevant low-degree maps on the E_2 page. The relevant map on E_3 is a Bockstein β, which is seen to be the zero map by examining the long exact sequence in which it is defined:

$$\cdots \longrightarrow H^2(Q, \mathbb{F}_p) \longrightarrow H^2(Q, \mathbb{Z}/p^2) \longrightarrow H^2(Q, \mathbb{F}_p) \xrightarrow{\beta} H^3(Q, \mathbb{F}_p) \longrightarrow \cdots$$
$$\cong \qquad\qquad \cong \qquad\qquad \cong \qquad\qquad \cong$$
$$(\mathbb{Z}/p)^3 \qquad (\mathbb{Z}/p^2)^3 \qquad (\mathbb{Z}/p)^3 \qquad (\mathbb{Z}/p)^4$$

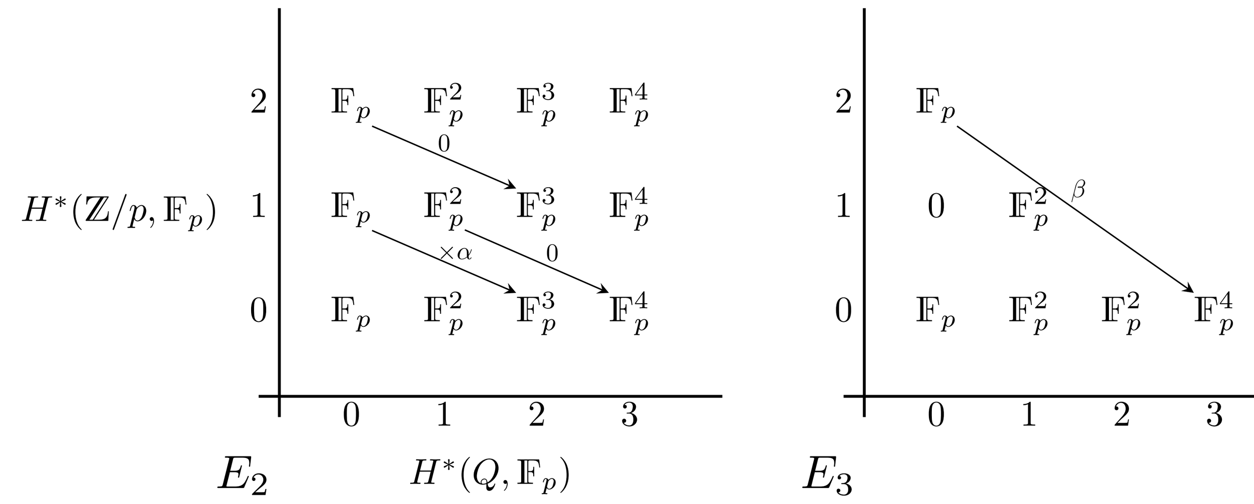

Figure: The E_2 and E_3 pages of the LHS spectral sequence for $H^*(C, \mathbb{F}_p)$

Let G and K be efficient finite p-groups. Then $G \times K$ is efficient.

By a theorem of Rapaport, G and K have efficient presentations with $d(G)$ and $d(K)$ generators. Then the Künneth formula

$$H_2(G \times K, \mathbb{Z}) \cong H_2(G, \mathbb{Z}) \oplus (H_1(G, \mathbb{Z}) \otimes H_1(K, \mathbb{Z})) \oplus H_2(K, \mathbb{Z})$$

implies that the obvious presentation for $G \times K$ is efficient.

NB: If G and K are not p-groups, there is no reason to hope that $d(H_1(G, \mathbb{Z}) \otimes H_1(K, \mathbb{Z})) = d(G) \times d(K)$; indeed, it could be zero!

Careful counting lets us prove the following by a cunning induction.

Theorem [G17]

Let n be a negative integer and p a prime. There exist natural numbers q, r, and s such that the finite p-group $A^q \times B^r \times C^s$ has deficiency n.

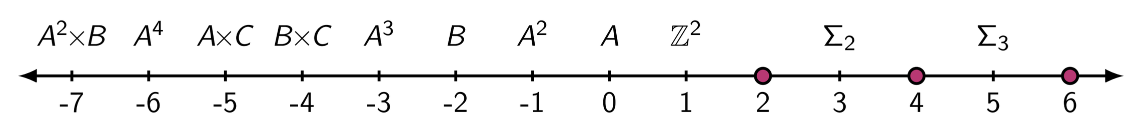

Figure: Deficiencies of Kähler groups

G. Gardam, *Finite p-groups of arbitrary deficiency*, preprint (2017).

D. Kotschick, *The deficiencies of Kähler groups*, J. Topol. **5** (2012), no. 3, 639–650. MR 2971609