Deficiencies of finite groups

Giles Gardam

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Let G be a finitely presented gp.
Q$^{\text{n}}$. What is the smallest pres$^{\text{n}}$ of G? (E.g. fewest relators.)
E.g. $\mathbb{Z}^n \cong \langle\, a_1, \ldots, a_n \mid [a_i, a_j], i < j \,\rangle$

Can we do better than $\binom{n}{2}$ rels?
Def$^{\text{n}}$. *Deficiency*

$$\operatorname{def}(G) = \max_{|\langle X \mid R \rangle| \cong G} (|X| - |R|).$$

Why is max defined?

$$\operatorname{def}(G) \leq \operatorname{rk}(H_1(G)) - d(H_2(G))$$

$G_{ab} \cong \mathbb{Z}^{\text{rk}} \oplus \text{Torsion}$

If equality, call G *efficient*.
Thm (B. BAUMSLAG–PRIDE, '78)**.** $\operatorname{def}(G) \geq 2 \implies G$ is large

$H_2(\mathbb{Z}^2) \cong \mathbb{Z}^{\binom{n}{2}} \implies \operatorname{def}(\mathbb{Z}^n) = n - \binom{n}{2}$.

$H_2(\Sigma_g) = \mathbb{Z} \implies \operatorname{def}(\Sigma_g) = 2g - 1$.

Standard pres$^{\text{n}}$ of RAAG is efficient, so

$$\operatorname{def}(A_\Gamma) = \chi(\Gamma)$$

$\implies \operatorname{def}(G)$ can be any integer!

What about finite gps?
Motivation (KOTSCHICK, '12)**.** $\operatorname{def}(\text{ Kähler gp}) > 0 \implies$ odd. All negative?

. Life is cruel! Not all gps are efficient.

Thm (SWAN, '65)**.** $G_k := (\mathbb{Z}/7)^k \rtimes_{\times 2} \mathbb{Z}/3$ has $H_2(G_k) \cong 0$ but $\operatorname{def}(G_k) \to -\infty$.
Thm (LUSTIG, '95)**.** $B_3 \times \mathbb{Z}$ is not efficient.
Lem. G, H efficient finite p-gps $\implies G \times H$ efficient, and

$$\operatorname{def}(G \times H) = \operatorname{def}(G) + \operatorname{def}(H) + d(G)d(H)$$

$$\begin{aligned} A_p &:= \langle\, a, b \mid a^p = b^p = [a, b] \,\rangle \\ B_p &:= \langle\, a, b \mid a^p,\ b^p,\ [[a, b], a],\ [[a, b], b] \,\rangle \\ C_p &:= \langle\, a \mid a^p \,\rangle \end{aligned}$$

Thm (G, '17)**.** $\forall\, p, n, \quad \exists\, r, s, t$ s.t.

$$\operatorname{def}(A_p^r \times B_p^s \times C_p^t) = -n.$$