Encoding and Detecting Properties in Finitely Presented Groups

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Acknowledgements

> «Ach was...», fauche ich. «Aber kuck dir die Bilder mal an! Da hat doch garantiert ein Erwachsener geholfen.» «Ja, das ist wirklich gemein», sagt das Känguru. «Wenn dir ein Erwachsener geholfen hätte, wäre dein Buch bestimmt auch besser geworden.»
>
> Marc-Uwe Kling, *Die Känguru-Offenbarung*

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Abstract

In this thesis we study several properties of finitely presented groups, through the unifying paradigm of encoding sought-after group properties into presentations and detecting group properties from presentations, in the context of Geometric Group Theory.

A group law is said to be detectable in power subgroups if, for all coprime m and n, a group G satisfies the law if and only if the power subgroups $G^{(m)}$ and $G^{(n)}$ both satisfy the law. We prove that for all positive integers c, nilpotency of class at most c is detectable in power subgroups, as is the k-Engel law for k at most 4. In contrast, detectability in power subgroups fails for solvability of given derived length: we construct a finite group W such that $W^{(2)}$ and $W^{(3)}$ are metabelian but W has derived length 3. We analyse the complexity of the detectability of commutativity in power subgroups, in terms of finite presentations that encode a proof of the result.

We construct a census of two-generator one-relator groups of relator length at most 9, with complete determination of isomorphism type, and verify a conjecture regarding conditions under which such groups are automatic. Furthermore, we introduce a family of one-relator groups and classify which of them act properly cocompactly on complete CAT(0) spaces; the non-CAT(0) examples are counterexamples to a variation on the aforementioned conjecture. For a subclass, we establish automaticity, which is needed for the census.

The deficiency of a group is the maximum over all presentations for that group of the number of generators minus the number of relators. Every finite group has non-positive deficiency. For every prime p we construct finite p-groups of arbitrary negative deficiency, and thereby complete Kotschick's proposed classification of the integers which are deficiencies of Kähler groups. We explore variations and embellishments of our basic construction, which require subtle Schur multiplier computations,

and we investigate the conditions on inputs to the construction that are necessary for success.

A well-known question asks whether any two non-isometric finite volume hyperbolic 3-manifolds are distinguished from each other by the finite quotients of their fundamental groups. At present, this has been proved only when one of the manifolds is a once-punctured torus bundle over the circle. We give substantial computational evidence in support of a positive answer, by showing that no two manifolds in the SnapPea census of 72 942 finite volume hyperbolic 3-manifolds have the same finite quotients.

We determine examples of sizeable graphs, as required to construct finitely presented non-hyperbolic subgroups of hyperbolic groups, which have the fewest vertices possible modulo mild topological assumptions.

Statement of Originality

I declare that the work in this thesis is, to the best of my knowledge, original and my own work, except where otherwise indicated, cited, or commonly known.

Chapter 1 reviews the literature and some observations that are well known to experts. The only original contribution in this chapter – beyond the exposition – is relating the theorem of [Rhe81] to a putative strategy for proving a group has a relation gap, which arose in a collaboration with Nicolaus Heuer.

Elsewhere in the thesis, examples were provided by Derek Holt (Remark 4.5) and Gareth Wilkes (Example 5.3), and the proof of Lemma 4.18 was suggested by Jakub Konieczny.

The material of Chapter 2 has appeared in the preprint [Gar17a], and some of Chapter 4 is the subject of the preprint [Gar17b].

This thesis has not been submitted for a degree at another university.

Giles Gardam

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Chapter 1

Introduction

1.1 Overview

This thesis is about finitely presented groups. The chapter titles, read in isolation, might suggest that a selection of disjoint topics in geometric, combinatorial, and classical group theory has been treated. However, they are linked by recurring paradigms, motifs, techniques, and lead characters.

The two central paradigms are the eponymous *encoding* and *detecting*. The main channel of encoding in this thesis is that of group presentations which we strive to make concise in an appropriate sense – sometimes optimally so – within a class of presentations of a fixed group: minimizing the number of relators, for example, minimizing the number of relators minus the number of generators, or both. This is the focus of Chapter 4 and Section 2.5 in Chapter 2. Having small presentations is also important from a practical point of view in Chapter 5. In Chapter 6 we explore how small we can make certain graphs such that the finitely presented groups that are determined by them still have certain geometric and topological properties; we consider this a type of encoding as well.

By *detecting* we mostly mean detecting, from a finite presentation, information about a group that is encoded in its invariants. These invariants are often computable, or at least partially computable. An example of such an invariant is the *profinite completion*, which features heavily in Chapter 5, but also in Chapter 3. The objective of Chapter 2 is to understand when one can detect if group laws hold by examining a more algebraic – and much less algorithmic – invariant, namely power subgroups. Nonetheless, there is an interesting range of algorithmic questions that arise in this context, in particular concerning concise presentations.

Alongside the profinite completion, some other recurring motifs are *CAT(0) geometry* and *finiteness properties*. Certain groups reappear multiple times; some specific lead characters are the *torus knot complement groups*. These are amalgamated free products of infinite cyclic groups and along with Baumslag–Solitar groups they lay claim to being (the fundamental groups of) the simplest graphs of infinite groups. They have a part to play in Chapters 3, 4, and 5.

Computation is a key paradigm that accompanies us throughout the thesis, whether in plain sight or not. We often tread the line between what is computable in theory and what is computable in practice; what is computable in theory need not be computable in practice, and we often make computations in practice which in theory need not succeed.

The structure of this thesis

Each chapter begins with its own comprehensive introduction. This introductory chapter gives a broad stroke overview, summarizes the main results, sets the scene for the rest of the thesis, covers some background material (most of which would be familiar to experts) that will be referred to later, and discusses the relation gap problem.

In particular, non-positively curved groups recur in Chapter 3 (and tacitly in Chapter 5). Questions around non-positively curved and hyperbolic groups motivate Chapter 6. As signalled above, computing with the profinite completion features in Chapter 3 and features heavily in Chapter 5. Other computational ideas resurface in Chapter 2, particularly in Section 2.5.

The chapters can be read independently of each other, with the exception of their reference to standard background material in this chapter and occasional clearly marked inter-referencing.

1.2 Summary of results

Chapter 2. Detecting laws in power subgroups

This chapter studies the following broad question: what can be deduced about a group G by examining its power subgroups $G^{(n)} = \langle g^n : g \in G \rangle$? In particular, can one infer which laws – for instance nilpotency or solvability – G satisfies? We say a law is detectable in power subgroups if, for all coprime m and n, a group G satisfies the law if and only if the power subgroups $G^{(m)}$ and $G^{(n)}$ both satisfy the law (this choice of definition is discussed in Section 2.2.2). The starting point for this chapter was a folk theorem, recently rediscovered in [Ven16], that commutativity is detectable in power subgroups, which we generalize as follows:

Corollary A1. *Let m and n be coprime and let $c \geq 1$. Then a group G is nilpotent of class at most c if and only if $G^{(m)}$ and $G^{(n)}$ are both nilpotent of class at most c.*

Detectability of laws in power subgroups admits an elegant phrasing in the language of varieties of groups (which we develop in Section 2.2.1). We write $\mathcal{B}_m$ for the Burnside variety of groups of exponent m. The above corollary is a consequence of the following theorem:

Theorem A. *Let $\mathcal{V}$ be a locally nilpotent variety and let m and n be coprime. Then*

$$\mathcal{V}\mathcal{B}_m \cap \mathcal{V}\mathcal{B}_n = \mathcal{V}.$$

We show that this cannot be generalized from the nilpotent setting to the solvable setting, since in particular we have:

Theorem B. *There exists a finite group W such that $W^{(2)}$ and $W^{(3)}$ are both metabelian but W is of derived length 3.*

The smallest order of such a group W is 1458.

Detectability of laws naturally raises interesting questions about concise finite presentations, which essentially ask how many instances of a law we must witness in power subgroups in order to conclude that it holds in the whole group. We analyse the complexity of the abelian case, and show that one needs surprisingly few test elements:

Theorem C. *Let m and n be coprime. The following is a presentation of $\mathbb{Z} \times \mathbb{Z}$:*

$$\langle\, a, b \mid [a^m, b^m], [a^m, (ab)^m], [b^m, (ab)^m], [a^n, b^n], [a^n, (ab)^n], [b^n, (ab)^n] \,\rangle.$$

Chapter 3. A census of small two-generator one-relator groups

A one-relator group is a group defined by a presentation with a single relator. It is a classical theorem of Magnus that the word problem – the task of taking a word in a fixed generating set of a group and deciding whether or not that word represents the identity element – is solvable for every one-relator group [Mag32]. Motivated by questions around the specific complexity of the word problem for one-relator groups, we construct via computer and manual work a census of two-generator one-relator groups of relator length at most 9. The census is listed in Appendix A.

We will formally introduce automatic groups, CAT(0) groups, and Baumslag–Solitar groups in Section 1.3. The Baumslag–Solitar groups are (in certain settings) an obstruction to the "non-positive curvature" of automatic and CAT(0) groups. One is lead to ask: when is this the only obstruction?

Problem ([MUW11, Problem 1.5])**.** Is it true that every one-relator group with no unbalanced Baumslag–Solitar subgroup is automatic?

This problem is in the style of the so-called Gersten Conjecture, Question 1.1 below, but with negative curvature replaced by non-positive curvature. We answer this problem for our census groups.

Theorem D. *Every two-generator one-relator group $\langle\, a, b \mid r \,\rangle$ of relator length $|r| \leq 9$ is either automatic or has an unbalanced Baumslag–Solitar subgroup.*

We completely determine the 134 isomorphism types, noting:

Theorem E. *The isomorphism classes of two-generator one-relator groups with relator length at most 9 are determined by automorphic orbit and inversion of the relator, except for 6 exceptional pairs.*

The well-known one-relator group $G_{1,2}$ introduced by Baumslag [Bau69] with all finite quotients cyclic is the only exception to "profinite rigidity" amongst these groups:

Theorem F. *Let G and H be defined by two-generator one-relator presentations with relator length at most 9. Suppose that $\widehat{G} \cong \widehat{H}$. Then either $G \cong H$, or G and H are isomorphic to $\mathbb{Z}$ and $G_{1,2}$.*

We introduce a family of one-relator groups $R(p, q, k, l)$ (Definition 3.15) and classify which of them act properly cocompactly on complete CAT(0) spaces:

Theorem G. *The group $R(p, q, k, l)$ acts properly and cocompactly on a complete CAT(0) space if and only if $|k| > |l + \frac{p}{q}|$.*

We prove some of the groups $R(p, q, k, l)$ to be automatic in Theorem 3.19, which was a necessary step in the proof of Theorem D. We give a negative answer to a variation on the above problem from [MUW11], where "automatic" is replaced by "CAT(0)"; this variation is another way that we could translate the Gersten Conjecture from the setting of hyperbolic groups to non-positively curved groups.

Corollary G′. *Let k, l, p and q be integers as in Definition 3.15. Suppose that $|k| \leq |l + \frac{p}{q}|$. Then $R(p, q, k, l)$ is a one-relator group containing no unbalanced Baumslag–Solitar subgroup, and it does* not *act properly cocompactly on a complete CAT(0) space.*

Chapter 4. Finite p-groups of arbitrary negative deficiency

It is a very difficult problem in general to take a finitely presented group and determine how concise exactly one can make its finite presentations; this is related to problems in low dimensional topology such as the D(2) conjecture (see [BT07]). One particular measure of conciseness is deficiency. The deficiency of a group presentation is the number of generators minus the number of relators, and the deficiency of a group is the maximum deficiency over all its presentations. Any group of positive deficiency has infinite abelianization, so every finite group has non-positive deficiency.

Kotschick studied deficiencies of fundamental groups of compact Kähler manifolds, and determined that they cannot be positive even integers [Kot12]. The surface group of genus g is a Kähler group of deficiency $2g-1$, so this leaves only the question of which negative integers can be the deficiency of a Kähler group. Kotschick gave examples for almost all negative integers, and suggested that one should be able to obtain the rest using finite groups. We do this, and more.

Theorem H. *Let p be a prime and let A_p, B_p and C_p be the finite p-groups of Definition 4.6. Then for every $n \in \mathbb{N}$ there exist natural numbers r, s and t such that the finite p-group $A_p^r \times B_p^s \times C_p^t$ has deficiency $-n$.*

The proof works by considering a class $\mathcal{G}_p$ of finite p-groups in which deficiency is well-behaved. We then carefully analyse the (elementary) combinatorics and number theory of how deficiency changes with the powers r, s and t in such a direct product to ensure that we obtain all negative integers.

We then explore variations and embellishments of our basic construction in Sections 4.5 and 4.6, considering inputs that are structurally different: the numbers of generators and relators in "minimal presentations" vary, and the groups can be infinite. This requires subtle Schur multiplier computations, which are the subject of Section 4.8. We investigate the conditions on inputs to the construction that are necessary for success; this greater understanding makes the construction more transparent, but does not make our proofs of statements such as Theorem H redundant. Specifically we give a necessary condition, and a condition that is sufficient to obtain all but finitely many negative integers. For the sake of completeness in demonstrating that 'minimality' (in the sense of Definition 4.1) and 'efficiency' of presentations are orthogonal concepts, in Section 4.7 we note that Lustig's non-efficient torsion-free group admits a minimal presentation.

Chapter 5. Profinite rigidity in the SnapPea census

A well-known question asks whether any two non-isometric finite volume hyperbolic 3-manifolds are distinguished from each other by the finite quotients of their fundamental groups. At present, this has been proved only when one of the manifolds is a once-punctured torus bundle over the circle.

There is a naive partial algorithm that will succeed in proving that two groups have non-isomorphic profinite completions when this is indeed the case: enumerate finite quotients until a group is found that is a quotient of one but not the other. Such a naive procedure is very slow in practice, and is totally infeasible for working with large collections of groups. By a more informed approach, applying some theory of hyperbolic 3-manifolds and of profinite completions, we succeeded in verifying the following theorem computationally using 64 hours of CPU time.

Theorem I. *The 72 942 finite volume hyperbolic 3-manifolds in the SnapPea census are distinguished from each other by the finite quotients of their fundamental groups.*

This is the first wholesale evidence for a positive answer to the question, and provides independent verification – using only discrete methods – that there are no duplicates in the census.

Chapter 6. Minimal sizeable graphs

We determine examples of sizeable graphs (see Definition 6.1), as required to construct finitely presented non-hyperbolic subgroups of hyperbolic groups, which have the fewest vertices possible modulo mild topological assumptions.

Theorem J. *The smallest sizeable graph with each of the 4 defining subgraphs a path has 31 vertices.*

1.3 Non-positively curved groups

One of the most classical topics in geometric group theory is hyperbolic, or negatively curved, groups. A reflection of the fundamental importance of this class of groups is that it admits many different definitions, for instance δ-hyperbolicity, linear isoperimetric inequality, and admitting a "Dehn algorithm" to solve the word problem in linear time. There are multiple candidates for the correct notion of a non-positively curved group. In this section we survey the foremost of these classes of groups, their relationships, and open problems.

A famous question – sometimes called the Gersten Conjecture – is the following.

Question 1.1 (Q1.1 in [Bes04])**.** Let G be a group that admits a compact $K(G,1)$. If G does not contain any Baumslag–Solitar subgroups $BS(m,n)$ (see Definition 1.7), is G necessarily hyperbolic?

Baumslag–Solitar subgroups are a principal obstruction to negative or non-positive curvature. The conjecture asks whether their presence and the lack of a topological finiteness property are the only two obstructions. For more on the Baumslag–Solitar groups, see Section 1.3.6 below. Rather than assuming G has no Baumslag–Solitar subgroups, we could make the stronger assumption that G embeds in a hyperbolic group; it is also open whether this suffices for hyperbolicity (taken together with admitting a compact $K(G,1)$).

We define the torus knot group $T_{m,n} = \langle\, x, y \,|\, x^m = y^n \,\rangle$. When m and n are coprime, this is the fundamental group of the complement of the (m,n)-torus knot. Torus knot groups fall into each of the 3 classes of groups we will now introduce: CAT(0) groups, automatic groups, and free-by-cyclic groups. We note here that every torus knot group is virtually a direct product of $\mathbb{Z}$ and a free group, so they are all commensurable (this also makes automaticity immediate, but it is not known whether being CAT(0) is a commensurability invariant). The quotient of $T_{m,n}$ by the centre $\langle x^m \rangle$ is $\mathbb{Z}_m * \mathbb{Z}_n$, which has a free subgroup of finite index that we pull back to get a finite index subgroup whose quotient by a central $\mathbb{Z}$ is free. The short exact sequence must split, since the quotient is free. Being split and central, it is a direct product.

1.3.1 CAT(0) groups

A geodesic metric space is called *CAT(0)* if its geodesic triangles are no fatter than those of Euclidean space (for a precise definition, see [BH99, p. 158]). This is a metric form of non-positive curvature and it implies contractiblity. A geodesic metric space in which every point has a CAT(0) neighbourhood is called *locally CAT(0)* or simply *non-positively curved*.

A group acting properly and cocompactly by isometries on a CAT(0) space is called a *CAT(0) group*. The fundamental group of a compact locally CAT(0) space is CAT(0).

A key theorem is the Link Condition [BH99, II.5.2, p. 206] stated by Gromov, which says in particular that a Euclidean polygonal complex with finitely many isometry types of cells is CAT(0) if and only if the link of every vertex is CAT(1).

Thus, we can check the CAT(0) condition locally. For cube complexes the link condition is purely combinatorial, and for 2-dimensional complexes it is equivalent to there being no vertex link containing an embedded loop of length less than 2π.

There are many consequences of a group being CAT(0), which include

- solvable word and conjugacy problems (see Section 1.4.1);
- centralizers virtually split, that is, the cyclic subgroup generated by a group element of infinite order is virtually a direct factor in its centralizer;
- quadratic Dehn function (see Definition 1.9); *and*
- type F_∞ (see Section 1.3.4 below).

A torus knot group is the fundamental group of a locally CAT(0) space: take two circles of length m and n, and wrap the two ends of a cylinder of circumference mn around the loops (m times around one, and n times around the other). The links are all of the form of a pair of vertices joined by m or n edges of length π, which corresponds to how the edges look like the spine of a book, with multiple sheets hanging off. Seen another way, the universal cover is the direct product of a tree and a line, with are both CAT(0) spaces, so the universal cover is too.

1.3.2 Automatic groups

The topic of automatic groups emerged from conversations between Jim Cannon and Bill Thurston around algorithmic properties of Kleinian groups. The canonical reference is the book [ECH+92].

Let G be a group with a finite generating set A. A *combing* or *normal form* on G is a choice $\sigma(g)$ of word in the letters $A^{\pm 1}$ representing each element $g \in G$. That is, it is a map $G \to (A \cup A^{-1})^*$ which is a section of the monoid homomorphism from the free monoid $(A \cup A^{-1})^* \to G$. The word $\sigma(g)$ gives a path in the Cayley graph of G from 1 to g.

Definition 1.2. A combing σ has the *fellow traveller property* if there exists a constant $k > 0$ such that for all $g_1, g_2 \in G$,

$$d(\sigma_{g_1}(t), \sigma_{g_2}(t)) \leq kd(g_1, g_2)$$

for all $0 \leq t \leq \max\{|\sigma_{g_1}|, |\sigma_{g_2}|\}$, with distances measured in the word metric.

Definition 1.3. A group is *automatic* if it admits a combing σ such that

- σ has the fellow traveller property; *and*
- the formal language $\sigma(G) \subset (A \cup A^{-1})^*$ is a regular language.

That is, there is a finite state automaton (roughly, a dumb computer with finite memory) which recognizes the words of the combing. One can also define automatic groups via finite state automata that recognize group multiplication rather than the fellow traveller property.

Many groups of geometric interest are automatic, as we shall see in Section 1.3.5. Some important properties of automatic groups are

- word problem solvable in quadratic time (see Section 1.4.1);
- quadratic Dehn function (see Definition 1.9);
- automaticity is a commensurability invariant; *and*
- type F_∞ (see Section 1.3.4 below).

Since automaticity is a commensurability invariant and is closed under direct products (and free groups are automatic), torus knot groups are automatic. Other ways of proving this are listed on page 48.

1.3.3 Free-by-cyclic groups

A free-by-cyclic group is the extension of a finitely generated free group by $\mathbb{Z}$, or is equivalently the mapping torus $F \rtimes_\varphi \mathbb{Z}$ of an automorphism φ of a finitely generated free group F (this is equivalent since the short exact sequence $1 \to F \to G \to \mathbb{Z} \to 1$ must split). Free-by-cyclic groups enjoy many properties, including polynomial time solution to the word problem [Sch08], coherence [FH99] (that is, all finitely generated subgroups are finitely presented), and quadratic Dehn function [BG10]. However, they need not be CAT(0) or automatic.

Example 1.4 (Gersten, Proposition 2.1 in [Ger94]). Let $F_3 = F(a,b,c)$ and let $\varphi \in \operatorname{Aut}(F_3)$ be defined by

$$\begin{aligned} \varphi\colon a &\mapsto a \\ b &\mapsto ab \\ c &\mapsto a^2c. \end{aligned}$$

Then $G := F_3 \rtimes_\varphi \mathbb{Z}$ is not CAT(0).

A much more involved argument shows that this group is not automatic [BBR05].

To see that it is non-CAT(0), we change perspective a little, and recognize G as a tubular group. Specifically, it is a double HNN extension of $\mathbb{Z}^2$ over $\mathbb{Z}$ subgroups, namely

$$\langle\, a, t, b, c \mid [a, t],\, t^b = ta^{-1},\, t^c = ta^{-2} \,\rangle.$$

If G were to act properly and cocompactly on a CAT(0) space, the Flat Torus Theorem [BH99, II.7.1, p. 244] says that $\langle a, t\rangle \cong \mathbb{Z}^2$ would preserve a subspace isometric to the Euclidean plane on which it would act by translation (achieving the minimum translation length $|g|$ for each element g of $\langle a, t\rangle$). But conjugation by b and c implies that $|t| = |ta^{-1}| = |ta^{-2}|$. After fixing coordinates, this gives three collinear vectors in $\mathbb{Z}^2 \otimes \mathbb{R}$ with the same norm, which is impossible unless two were to coincide, which would make a elliptic, a contradiction.

It was proved by Brinkmann [Bri00] that a free-by-cyclic group is hyperbolic if and only if the free group automorphism φ is *atoroidal*, meaning that it has no periodic conjugacy class (for all $n \geq 1$ and $x \in F$, the element $\varphi^n(x)$ is *not* conjugate to x). Since every automorphism of $F_2 = F(a, b)$ sends $[a, b]$ to a conjugate of $[a, b]^{\pm 1}$ (this can be easily checked for Nielsen automorphisms, which generate $\mathrm{Aut}(F_n)$), no group $F_2 \rtimes \mathbb{Z}$ is hyperbolic.

Remark 1.5. A theorem of Bieri [Bie76b] states that every non-trivial infinite-index finitely presented normal subgroup of a group of cohomological dimension 2 is free. It is an open problem whether one-relator groups are coherent, that is, whether all their finitely generated subgroups are finitely presented; a homological version of coherence has recently been proved by Louder and Wilton [LW17] using ideas of Duncan and Howie [DH91]. Despite coherence being as yet unresolved, if the kernel of a surjection from a torsion-free one-relator group to $\mathbb{Z}$ is finitely generated then it is in fact free, and there is an elegant algorithm to decide whether this is the case [Bro87, § 4]. For reasons of Euler characteristic, only the two-generator case is relevant, and Brown's algorithm also deals with the case that the relator is in the commutator subgroup (and thus there are essentially distinct maps to $\mathbb{Z}$) and determines whether a two-generator one-relator group is free-by-cyclic or more generally an ascending HNN extension of a free group (that is, the mapping torus of an injective free group endomorphism). The reader is referred to [DT06] for an excellent exposition of Brown's algorithm.

One way to see that torus knot groups are free-by-cyclic is to apply Brown's criterion.

1.3.4 Finiteness properties

For a thorough introduction to topological finiteness properties, the reader is referred to [Geo08, p. 169ff].

Definition 1.6. A group G is said to be of *type* F_n if there is a $K(G, 1)$ that is a CW-complex with finite n-skeleton. We say G is of *type* F_∞ if it has a $K(G, 1)$ with only finitely many cells in each dimension. If G moreover has a $K(G, 1)$ with only finitely many cells, we say it is of *type* F.

Hyperbolic groups, CAT(0) groups, and automatic groups are all of type F_∞.

1.3.5 Surveying the landscape

We survey in Figure 1.1 the relations between the following classes of groups: CAT(0) groups, automatic groups, hyperbolic groups, groups acting properly and cocompactly on CAT(0) cube complexes (denoted CCC), and free-by-cyclic groups. While there are competing notions of "non-positively curved" group, a common criterion that all these classes fulfill is having at most quadratic Dehn function ("satisfying a quadratic isoperimetric inequality"). Some prominent classes that cluster around the notion of non-positive curvature but that we omit here are the relatively hyperbolic, semihyperbolic, and combable groups; for details, the reader is referred to [Bri06]. The notion of an acylindrically hyperbolic group [Osi16] captures a form of large-scale hyperbolic-like behaviour, but does not give any restrictions on subgroups, finiteness properties or Dehn functions (in particular, every non-elementary free product is acylindrically hyperbolic).

To check that Figure 1.1 faithfully represents the relationships between the five classes, consider constructing it as follows. First we draw the automatic groups, CAT(0) groups, and free-by-cyclic groups. As far as we know, there are no containment relations between these classes (although it is possible that all CAT(0) groups are automatic). Now we add the CCC groups, which are necessarily automatic [NR98] and CAT(0). Finally, the hyperbolic groups are all automatic, and any hyperbolic free-by-cyclic group is CCC [HW15], so of the 6 regions into which the automatic groups have thus far been carved, 4 contain hyperbolic groups.

We should note that the relative sizes of the regions in the diagram are not intended to convey the relative importance of the various classes of groups! Moreover, it is not intended to suggest that CCC groups are in some sense "closer" to CAT(0) groups than to automatic groups. Also, the distinction between whether a collection

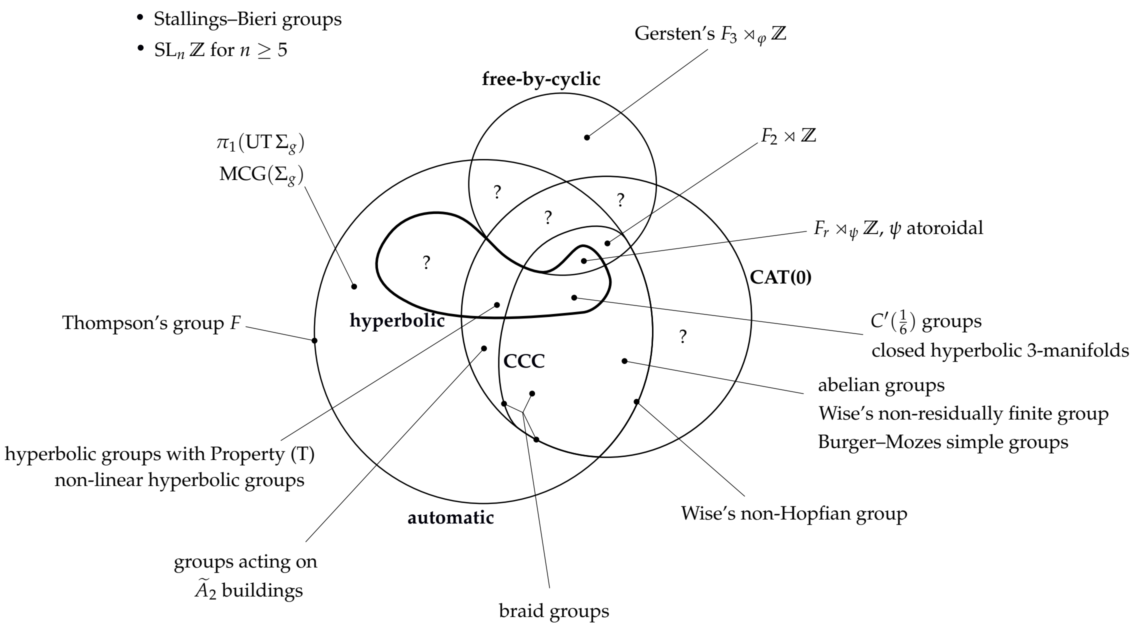


Figure 1.1: Various classes of groups satisfying a quadratic isoperimetric inequality: hyperbolic, CAT(0), CAT(0) cubulated, automatic, and free-by-cyclic

of groups is seen as a "class" or whether it is seen as an "example" is a nebulous one, and we again do not intend to diminish a topic of study by this labelling.

We do not know any examples of free-by-cyclic groups which fail to be one of automatic and CAT(0) but not the other; in general, we have poor knowledge of which free-by-cyclic groups are automatic or CAT(0) [BV06, Question 25]. We also know of no free-by-cyclic groups that are CAT(0) but cannot be cubulated. It is an open question whether every CAT(0) group is automatic, and whether every hyperbolic group is CAT(0).

The examples of groups given in Figure 1.1, in clockwise order, are the following.

- Gersten's Example 1.4 $F_3 \rtimes \mathbb{Z}$ is neither CAT(0) (as described above) nor automatic (by [BBR05]).

- A group $F_2 \rtimes_\varphi \mathbb{Z}$ cannot be hyperbolic, but it is cubulated by Button and R. Kropholler [BK16].

- $F_r \rtimes_\varphi \mathbb{Z}$ with φ atoroidal (so necessarily $r \geq 3$) is hyperbolic and is cubulated by [HW15].

- Many hyperbolic groups are now known to be cubulated, including all $C'(\frac{1}{6})$ small cancellation groups and fundamental groups of all closed hyperbolic 3-manifolds.

- Among the CAT(0) automatic groups which are not hyperbolic are (finitely generated) abelian groups, as well as pathological groups such as the Burger–Mozes simple groups [BM00] and Wise's non-Hopfian group [Wis96b]. It is unknown whether a CAT(0) group with no subgroup isomorphic to $\mathbb{Z}^2$ is necessarily hyperbolic.

- It is still open whether the non-Hopfian CAT(0) group constructed by Wise is automatic [Wis96a].

- The braid groups are automatic, and B_3 is cubulated (it is in fact isomorphic to the trefoil knot complement group $T_{2,3}$). For B_4, B_5 and B_6, the group is known to be CAT(0) [BM10; HKS16], but not known to be cubulated.

- Groups acting properly and cocompactly on $\widetilde{A}_2$-buildings (which are CAT(0) spaces with subspaces isometric to the Euclidean plane, or "flat planes", whence non-hyperbolicity) have Property (T) [BdHV08, page 5.7.7], so by [NR97b] they cannot be cubulated.

- There are hyperbolic groups with property (T), for instance lattices in $\mathrm{Sp}(n,1)$ for $n \geq 2$, which therefore cannot be cubulated (again by [NR97b]).

  By the work of Agol and Wise, if a hyperbolic group is cubulated then it must enjoy many properties, including being large (having a finite index subgroup that maps onto a non-abelian free group) and being linear. The existence of non-linear hyperbolic groups was proved by M. Kapovich [Kap05].

- Thompson's group F cannot act properly cocompactly on a CAT(0) space (since it has infinite cohomological dimension), however it has type F_∞ [BG84], quadratic Dehn function [Gub06], and it is an open problem whether or not it is automatic.

- Classic examples showing that an automatic group need not be CAT(0) are mapping class groups (non-CAT(0) by [KL96] or [BH99, II.7.26, pp. 257–8] and automatic by [Mos95]) and the fundamental group of the unit tangent bundle to a hyperbolic surface (non-CAT(0) by noting that centralizers do not split [BH99, II.7.26, pp. 257–8], and automatic by [NR97a]).

- The Stallings–Bieri group SB_n ($n \geq 3$) is the kernel of the map $F_2^n \twoheadrightarrow \mathbb{Z}$ sending each free generator in each direct factor to a generator of $\mathbb{Z}$. It does not fit into any of the classes of groups considered since it is not of type F_∞: it is a standard example of a group of type F_{n-1} not F_n [Sta63] [Bie76a]. The Dehn function is however quadratic, by [DERY09] for $n = 3$ and [CF17] for the general case.

  Robert Young proved in [You13] that $\mathrm{SL}_n \mathbb{Z}$ has quadratic Dehn function for $n \geq 5$, verifying a conjecture of Thurston in all cases except $n = 4$, which remains open. (For $n = 3$ the Dehn function is exponential [ECH+92], and for $n = 2$ the group is virtually free.)

1.3.6 Pathologies

Definition 1.7. Let m and n be non-zero integers. The *Baumslag–Solitar group* $\mathrm{BS}(m,n)$ is defined by the presentation

$$\mathrm{BS}(m,n) := \langle\, a, t \mid (a^m)^t = a^n \,\rangle.$$

If $|m| = |n|$, we call the group $\mathrm{BS}(m,n)$ *balanced*, and otherwise we call it *unbalanced*.

(In general, in the terminology of [Wis00], a group is called unbalanced if it contains an element g^m which is conjugate to g^n for some integers m and n with $|m| \neq |n|$.)

Note that $\mathrm{BS}(m,n) \cong \mathrm{BS}(-m,-n)$, so we can assume $|n| \geq m > 0$.

Proposition 1.8. *Let $|n| \geq m > 0$. Then the Baumslag–Solitar group $\mathrm{BS}(m,n)$ is*

- *solvable if and only if $m = 1$, in which case it is metabelian;*
- *automatic and CAT(0) if and only if it is balanced (that is, $m = \pm n$), and otherwise it has exponential Dehn function.*

For details, see [Col94] and the references therein.

A Baumslag–Solitar group contains a distorted (that is, not quasi-isometrically embedded) cyclic subgroup if $|m| \neq |n|$, and therefore cannot be a subgroup of a CAT(0) or hyperbolic group. However, the picture is less clear for automatic groups. In geometric dimension 2, Gersten has shown that an automatic group cannot have such a Baumslag–Solitar subgroup [Ger92]. On the other hand, examples of [BBMS97] show that there exist biautomatic groups with subgroups that have exponential Dehn functions, thereby removing a likely obstruction.

1.4 Computation

In this section we give a broad overview of decision problems for groups, present some basic theory for computing with the profinite completion, and discuss a putative computational approach to the "relation gap problem".

1.4.1 Degrees of decidability

A standard reference for decision problems in groups is the survey [Mil92]. Our intention here is to convey the main ideas, without getting lost in details (such as formally defining Turing machines).

Three fundamental decision problems for groups were proposed by Dehn in 1911: the *word problem*, the *conjugacy problem*, and the *isomorphism problem* [Deh11]. The word problem is the task of deciding, given a word in a fixed generating set of a group (and their inverses), whether or not that word represents the identity element. The conjugacy problem is the task of deciding, given words describing two group elements, whether or not they are conjugate. The isomorphism problem is the task of deciding, given two finite presentations of groups, whether they define isomorphic groups; this contrasts to the first two problems, which ask about elements in a single group.

The word problem was famously shown to be undecidable in the 1950s by Novikov and Boone: there exist groups for which there provably does not exist an algorithm (in any reasonable sense) to solve the word problem. The word problem reduces to the conjugacy problem, in the sense that if you can solve the conjugacy problem you can solve the word problem: simply ask whether or not the given element is conjugate to the identity. Thus the conjugacy problem is also undecidable. It was later shown that the conjugacy problem can be undecidable when the word problem is decidable, and that the isomorphism problem is undecidable.

Nonetheless, when one restricts to specific classes of groups of geometric interest (where one cannot embed the workings of an arbitrary Turing machine), these problems are often decidable. The word problem and conjugacy problem are decidable for hyperbolic and CAT(0) groups, and the word problem is solvable (in quadratic time) for automatic groups (where the conjugacy problem is open). Amongst hyperbolic groups, the isomorphism problem is decidable, by work of Sela [Sel95] and Dahmani–Guirardel [DG11].

A recurring theme in decision problems for groups is that of partially decidable problems, that is, problems where an algorithm exists that can at least answer

'YES' correctly. Each of Dehn's 3 problems is partially decidable. Suppose that $G = \langle A \mid R \rangle$ is a finitely presented group, and let $w \in F(A)$ be a word. If $w =_G 1$, then it can be written in $F(A)$ as

$$w = \prod_{i=1}^{n} u_i^{-1} r_{j_i}^{\epsilon_i} u_i \tag{$\clubsuit$}$$

for some positive integer n, $\epsilon_i \in \{\pm 1\}$, $r_{j_i} \in R = \{r_1, \ldots, r_m\}$ and $u_i \in F(A)$. The countably infinite set of such expressions can be naively enumerated, using 'diagonalization'. This use of the word diagonalization reflects the diagonalization argument used to show that $\mathbb{N}^2$ is countable. For instance, at the k-th iteration, we could enumerate all such products with $n \leq k$ and word lengths $|u_i| \leq k$; thus each iteration is a finite computation, and if such an expression for w exists we will find it in finite time (after freely reducing both words to decide equality in $F(A)$). Thus the word problem is at least partially decidable in every group: given a word that represents the identity, our naive algorithm can prove this in finite time. Given a word which does not represent the identity, this algorithm will of course never terminate. Likewise, one can show that a finite presentation of a group G can be transformed into any other finite presentation of the same group by a finite sequence of Tietze moves, so the isomorphism problem is partially decidable. For the conjugacy problem, we diagonalize the search for a conjugating element and the solution of the word problem. This diagonalization is a little more subtle than the previous one: as we increase the length of conjugating elements that we attempt, we also increase the number of computation steps for which we run each instance of the partial algorithm for the word problem.

A key point that we emphasize is that for an unsolvable decision problem that admits a partial algorithm, we cannot in any way predict how long such an algorithm will take. If we had some computable bound (in terms of the size of the input) on the maximum time the algorithm could run before successfully returning 'YES', then after this point we could halt the computation and return 'NO', which contradicts undecidability.

Decidability is an inherently theoretical concept: if a problem is decidable, that only means that there is an algorithm to solve it that will *eventually* terminate with one answer or another; the existence of such an algorithm may be of no use to us in practice. A problem is in P if there is an algorithm that solves it in time polynomial in the size of the input; we consider such algorithms to be fast. For instance, a hyperbolic group has word problem solvable in linear time [ABC+91, Theorem 2.18], and a linear group has polynomial time word problem [LZ77] (the key is encoding the matrix entries in polynomial space). Beyond P is the class EXP of problems admitting exponential time solutions, and then further classes where we still have *a priori* knowledge of how long an algorithm will take. For such problems, even if we have to wait a long time for an answer, we have a reasonable idea of just how long.

A common way in which we encounter decidable problems with algorithms of unbounded complexity is when combining two partial algorithms for a problem. For instance, the word problem is decidable in residually finite groups: by day we apply the standard partial algorithm for the word problem in an attempt to prove that w is trivial, and by night we enumerate finite quotients of the group in an attempt to find a quotient in which w survives. One of these procedures is guaranteed to succeed (and only one!), but we of course do not know which, and we also do not know how long it will take. (One conceptually simple way of enumerating maps to finite groups is to exhaustively try every way of sending the generators of the group to elements of the symmetric group S_n, for increasing n, and test that the images of the defining relators are all trivial.)

Another example of combining two partial algorithms is the decidability of hyperbolicity for free-by-cyclic groups. There is a partial algorithm taking any finite presentation that will succeed in proving the group to be hyperbolic if it is [Pap96]. For free-by-cyclic groups, assuming we can read the defining automorphism $\varphi \in \mathrm{Aut}(F_n)$ from the presentation (and if not, apply Tietze moves until we find such a presentation), we have a partial algorithm to prove non-hyperbolicity: try and find a non-trivial element $u \in F_n$ and $k \in \mathbb{Z}^+$ such that $\varphi^k(u)$ is conjugate to u.

A third example is deciding isomorphism in a "relatively profinitely rigid" class, as we will see in Proposition 1.15.

We note that, further to questions of complexity and the difficulty of implementation, there is a difference between the existence of algorithms and actually knowing the algorithm. Given a group and the abstract knowledge that it is linear does not enable us to solve its word problem in polynomial time: we need to know what the embedding in some $\mathrm{GL}_n \mathbb{C}$ actually is. Similarly, one needs a "Dehn presentation" of a hyperbolic group to solve its word problem in linear time, and although we can compute it from any given finite presentation of a hyperbolic group, there is no computable bound on how long this will take.

As mentioned, groups of geometric interest are often amenable to computation; the relationship between geometry and decision problems goes much further. For

$G = \langle A \mid R \rangle$, define the algebraic area $\mathrm{Area}(w)$ of a word $w \in F(A)$ such that $w =_G 1$ to be the smallest n such that one can write w as in (♣).

Definition 1.9. The *Dehn function* of $\langle A \mid R \rangle$ is defined as $\delta \colon \mathbb{N} \to \mathbb{N}$ by

$$\delta(n) = \max \{\mathrm{Area}(w) \mid w =_G 1, |w| \leq n\}.$$

Up to the appropriate asymptotic equivalence, this does not depend on the choice of presentation of G. The Dehn function describes the time needed for the naive partial algorithm to solve the word problem *on a non-deterministic Turing machine*, that is, if our algorithm were able to 'guess' the correct choices u_i, r_{j_i} and ϵ_i (a lemma on the diameters of van Kampen diagrams lets us give linear bounds on the lengths of the conjugating u_i).

The Dehn function records how many 2-cells, in the worst case, we need to fill a loop of length n in the Cayley complex of our group (the universal cover of its presentation complex, which has 1-skeleton the Cayley graph). Similarly, one can study isoperimetric inequalities for manifolds. Euclidean space has a quadratic isoperimetric function: the amount of soap needed to fill a wire loop of length ℓ is at most quadratic in ℓ. Geometry is intricately tied to decision problems in group theory via the Filling Theorem formulated by Gromov [Bri02, page 5.0.1]: the filling function of a manifold is equivalent to the Dehn function of its fundamental group. For a comprehensive introduction to Dehn functions, see [Bri02].

The Dehn function tells us when we can abandon the partial algorithm for the word problem, and conversely, a full solution to the word problem allows us to compute the Dehn function. Thus, a group has decidable word problem if and only if its Dehn function is bounded by a computable function, in which case it is computable. The Dehn function is a quasi-isometry invariant [Alo90], so solvability of the word problem is too.

The Dehn function only bounds the time complexity of one particular (non-deterministic) algorithm, but other algorithms can do much better. For instance, there are linear groups, even virtually special groups [Bri14, page 11.1], with exponential Dehn function. It is tempting then to think that there must be an enormous gulf between Dehn functions and clever algorithms that we can concoct. However, the celebrated theorem of Birget–Ol′shanskii–Rips–Sapir [BORS02] tells us that this is not so: a group has word problem in NP if and only if it embeds in a finitely presented group of polynomial Dehn function.

1.4.2 Computing with the profinite completion

We refer to [RZ10] for fundamental results on profinite completions.

Definition 1.10. Let G be a group. The *profinite completion* $\widehat{G}$ of G is the inverse limit

$$\varprojlim_{N \triangleleft G, [G:N]<\infty} G/N$$

of the inverse system of finite quotients of G.

The profinite completion has the expected universal property.

Lemma 1.11. *There is a natural map $\iota\colon G \to \widehat{G}$ such that every map from G to a finite group Q factors through $\widehat{G}$ uniquely.*

$$\begin{tikzcd} & \widehat{G} \arrow[d, dashed, "\exists!\Phi"] \\ G \arrow[ur, hook, "\iota"] \arrow[r, "\varphi"'] & Q \end{tikzcd}$$

The map ι is an embedding if and only if G is residually finite.

Proposition 1.12 ([RZ10, Corollary 3.2.8])**.** *Let G_1 and G_2 be finitely generated groups with the same finite quotients. Then $\widehat{G_1} \cong \widehat{G_2}$.*

Lemma 1.13 ([RZ10, Proposition 3.2.2])**.** *Let H be a finite index subgroup of G. Let $\overline{H}$ denote the closure of $\iota(H)$ in the profinite topology on $\widehat{G}$. Then $\overline{H} \cong \widehat{H}$, and the isomorphism is natural.*

Corollary 1.14. *There is a one-to-one correspondence between finite index subgroups of a finitely generated group G and finite index (open) subgroups of $\widehat{G}$. This bijection preserves profinite completion, and thus abelianization, as well as normality and the isomorphism class of quotients by normal subgroups.*

The word *open* can be omitted in the statement of the corollary, by the Nikolov–Segal Theorem [NS07] (for which finite generation of G is essential); this is not needed for our applications of this fact, since we only compare finite index subgroups of discrete groups whose profinite completions are isomorphic.

By combining the universal property, and Proposition 1.12, we see that the set of (isomorphism classes of) finite quotients of a finitely generated group G determines in particular the *number* of surjections of G onto any finite group.

Two standard approaches to proving a group G to be non-trivial are

- find a non-trivial finite group onto which G maps (or equivalently, a proper finite index subgroup); *and*
- show that G has non-trivial abelianization.

To show that a group is infinite, one combines these approaches and attempts to find a finite index subgroup of infinite abelianization.

These techniques are actually two sides of the same coin: both are attempting to compute invariants of the profinite completion $\widehat{G}$.

In contrast to the majority of decision problems for groups that one encounters, for profinite completions of finitely presented groups, distinguishing groups $\widehat{G}$ and $\widehat{H}$ is the easy direction: one can enumerate all maps from the groups to finite groups (for instance, via maps to finite symmetric groups of increasing degree) and wait until there is a finite group onto which G maps, say, but H doesn't.

The other direction is unsolvable: Bridson and Wilton proved that one cannot even decide if the profinite completion of a finitely presented group is trivial [BW15].

Suppose that one has a set S of groups which is *relatively profinitely rigid*; that is, for $G, H \in S$, we have $G \cong H$ if and only if $\widehat{G} \cong \widehat{H}$. Then given two finite presentations of groups in S we can decide whether the two groups are isomorphic: by day, we attempt to construct a proof that $G \cong H$, and by night, we attempt to prove that $\widehat{G} \not\cong \widehat{H}$. It is guaranteed that one of these procedures will terminate:

Proposition 1.15. *The isomorphism problem is solvable in a relatively profinitely rigid class of finitely presented groups.*

1.4.3 Relation Gap Problem

Let G be a finitely presented group. It is very difficult in general to say anything about the minimum number of relators needed to present G; this difficulty remains even after we fix a choice of generators for G.

Suppose that $G = \langle X \mid S \rangle$ and write $F = F(X)$ and $R = \langle\!\langle S \rangle\!\rangle_F$. These data give a short exact sequence

$$1 \to R \to F \to G \to 1.$$

We would like determine the minimum number of generators for R as a *normal* subgroup of F, which we write $d_F(R)$. (A well-known result says that if G is infinite, then R will be of infinite rank unless it is trivial). To give a very concrete example of how poorly we are able to deal with such questions, it is an open problem originating in a paper of Epstein [Eps61] whether the kernel of the presentation for $(\mathbb{Z} \times \mathbb{Z}/2) * (\mathbb{Z} \times \mathbb{Z}/3)$ as $\langle\, a, b, c, d \mid [a, b],\ b^2,\ [c, d],\ d^3 \,\rangle$ can be generated by 3 elements as a normal subgroup of $F(a, b, c, d)$.

The action of F on itself by conjugation restricts to an action on R. This then descends to an action on the abelianization R_{ab} (since the commutator subgroup is characteristic). Any element $r \in R$ will act trivially on R_{ab}, so the action factorizes through $F/R \cong G$, that is, R_{ab} has the structure of a G-module. This is the *relation module* for the presentation of G. A set of normal generators for $R \lhd F$ will give a set of generators for the relation module, so the rank of the relation module gives a lower bound on $d_F(R)$. (In the case of Epstein's example, the rank of the relation module is 3, in contrast to the best known presentation.)

It is expected that this crude bound cannot predict $d_F(R)$ exactly. If the number of relators needed does indeed exceed the rank of the relation module, then we say the group has a *relation gap*. While there are several examples of groups that are presumed to have a relation gap, there is not a single group for which it has been proved that this is the case. (The examples due to Bestvina and Brady [BB97] of groups which are of type FP_2 but are not finitely presented have what can only be called an *infinite* relation gap, but the relation gap problem *per se* is restricted to finitely presented groups.)

We now outline a reasonable line of attack towards proving that some putative example has a relation gap, and its fatal flaw. This unsuccessful approach, and the explanation of why it cannot succeed, arose in joint work with Nicolaus Heuer.

The relation module is introduced because we do not know how to compute $d_F(R)$, so we work instead in a tamer quotient (namely $F/[R, R]$). Meanwhile, the selection of groups available to us as quotients of F is unrestricted (other than by rank), so it is natural to try some different quotients. The finite groups suggest themselves as a sensible target, because if $\varphi \colon F \twoheadrightarrow E$ with E finite and $N := \varphi(R)$ we can naively compute $d_E(N)$ (if we are willing to wait long enough): simply enumerate all the k-element subsets of N, for increasing k, and determine whether they normally generate N.

However, a theorem of Rhemtulla – building on Kutzko's resolution [Kut76] of the Wiegold Problem for finite groups – shows that this cannot work.

Theorem 1.16 ([Rhe81])**.** *If N is a normal subgroup of a group E and E acts on N by conjugation, then $d_E(N) = d_E(N/N')$ provided $d_E(N)$ is finite and N has the following property:*

There does not exist an infinite descending series of E-subgroups $N' = C_0 > C_1 > \cdots$ with each C_i/C_{i+1} perfect.

For us, N/N' is an abelian quotient of R and thus a quotient of R_{ab}, this tells us that $d_E(N)$ is certainly no larger than the rank of the relation module, so cannot be used to prove that G has a relation gap.

$$\begin{tikzcd}
 & 1 \arrow[r] & R \arrow[r] \arrow[d] \arrow[dll] & F \arrow[r] \arrow[d] & G \arrow[r] & 1 \\
R_{ab} \arrow[d] & 1 \arrow[r] & N \arrow[r] \arrow[dll] & E \arrow[r] & E/N \arrow[r] & 1 \\
N_{ab} & & & & &
\end{tikzcd}$$

This theorem tells us more: we also could not hope to prove a relation gap by considering a solvable quotient of F (if E is solvable, then so is N, and no non-trivial C_i/C_{i+1} could be perfect). However, this does not immediately rule out taking a solvable quotient of R which need not extend to a solvable quotient of F.

Chapter 2

Detecting laws in power subgroups

2.1 Introduction

This chapter studies the following broad question: what can be deduced about a group G by examining its power subgroups $G^{(n)} = \langle g^n : g \in G \rangle$? In particular, can one infer which laws G satisfies?

Let $F_\infty = F(x_1, x_2, \dots)$ be the free group on the basis $\{x_1, x_2, \dots\}$. A *law* (or *identity*) is a word $w \in F_\infty$, and we say a group G *satisfies* the law w if $\varphi(w) = 1$ for all homomorphisms $\varphi \colon F_\infty \to G$. For notational convenience, when we require only variables x_1 and x_2 we will instead write x and y. We can also think of a law w on k variables $x_1, \dots, x_k$ as a function $w \colon G^k \to G$, written $w(g_1, \dots, g_k) := \varphi(w)$ for a homomorphism $\varphi \colon F_\infty \to G$ such that $\varphi(x_i) = g_i$.

Laws give a common framework for defining various group properties; basic examples include commutativity (corresponding to the law $[x, y]$), having exponent m (the Burnside law x^m), being metabelian (the law $[[x_1, x_2], [x_3, x_4]]$), and nilpotency of class at most c (the law $[[[\dots[x_1, x_2], x_3], \dots, x_c], x_{c+1}]$).

Definition 2.1. A group law w is *detectable in power subgroups* if, for all coprime m and n, a group G satisfies w if and only if the power subgroups $G^{(m)}$ and $G^{(n)}$ both satisfy w.

A subgroup of G will satisfy all the laws of G, but in general it is possible even for coprime m and n that the power subgroups $G^{(m)}$ and $G^{(n)}$ satisfy a common law that G does not; for example, the holomorph $G = \mathbb{Z}_7 \rtimes \mathbb{Z}_6$ (where $\mathbb{Z}_6 \cong \operatorname{Aut} \mathbb{Z}_7$ acts faithfully) was shown to have this property in [NNN62, Example 8.2] (and is in fact the smallest such group). A concrete example of a law that holds in $G^{(2)}$ and $G^{(3)}$ but not G is $[[x^2, y^2]^3, y^3]$. Another basic example is the holomorph $G = \mathbb{Z}_9 \rtimes \mathbb{Z}_6$, which does not satisfy the law $[x^2, x^y]$ although $G^{(2)}$ and $G^{(3)}$ do.

Example 2.2. The law x^r is detectable in power subgroups.

This basic example is immediate: for every $g \in G$, if $(g^m)^r = 1$ and $(g^n)^r = 1$, then $g^r = 1$ as m and n are coprime.

A classical theme in group theory is the study of conditions that imply that a group is abelian. This was recently revived by Venkataraman in [Ven16], where she proved that commutativity is detectable in power subgroups for finite groups. We can extend this to infinite groups using residual finiteness of metabelian groups (a theorem of P. Hall [Rob96, page 15.4.1]); it appears that this result is folklore.

In this chapter we prove that this result generalizes to the nilpotent case:

Corollary A1. *Let m and n be coprime and let $c \geq 1$. Then a group G is nilpotent of class at most c if and only if $G^{(m)}$ and $G^{(n)}$ are both nilpotent of class at most c.*

Fitting's Theorem (see 2.20 below) readily implies a weak form of the "if" direction, namely that G is nilpotent of class at most $2c$, but it is much less obvious that the precise nilpotency class is preserved.

Detectability of laws in power subgroups has an elegant formulation in the language of group varieties, which we develop in Section 2.2.1. The reader unfamiliar with varieties should not be deterred, as our use of this language is simply a means of expressing our reasoning in a natural and general setting. In particular, our treatment of varieties is essentially self-contained, and no deep theorems are called upon.

Let $\mathcal{N}_c$ denote the variety of nilpotent groups of class at most c and let $\mathcal{B}_m$ denote the 'Burnside' variety of groups of exponent m. Employing the notion of product varieties (Definition 2.8), we can restate the conclusion of Corollary A1 as $\mathcal{N}_c\mathcal{B}_m \cap \mathcal{N}_c\mathcal{B}_n = \mathcal{N}_c$. We prove this as a corollary of a stronger result:

Theorem A. *Let $\mathcal{V}$ be a locally nilpotent variety and let m and n be coprime. Then*

$$\mathcal{V}\mathcal{B}_m \cap \mathcal{V}\mathcal{B}_n = \mathcal{V}.$$

A variety is *locally nilpotent* if its finitely generated groups are nilpotent, or, equivalently, if its groups are locally nilpotent. A topic with a rich history, dating to work of Burnside, is that of Engel laws. The k-Engel law is defined recursively by $E_0(x,y) = x$ and $E_{k+1}(x,y) = [E_k(x,y), y]$. For example, the 3-Engel law is $[[[x,y],y],y]$. Havas and Vaughan-Lee [HV05] proved local nilpotency for 4-Engel groups, so we have the following:

Corollary A2. *Let m and n be coprime and let $k \leq 4$. A group G is k-Engel if and only if $G^{(m)}$ and $G^{(n)}$ are both k-Engel.*

It is an open question whether a k-Engel group must be locally nilpotent for $k \geq 5$. Recently A. Juhasz and E. Rips have announced that it does not have to be locally nilpotent for sufficiently large k.

The class of virtually nilpotent groups plays an important role in geometric group theory, dating back to Gromov's seminal Polynomial Growth Theorem: a finitely generated group is virtually nilpotent if and only if it has polynomial growth [Gro81]. Because of this prominence, we also prove that virtual nilpotency is detectable in power subgroups (Corollary 2.25).

In contrast, solvability of a given derived length is not detectable in power subgroups; this fails immediately and in a strong sense as soon as we move beyond derived length one, that is, beyond abelian groups.

Theorem B. *Let $\mathcal{M}$ denote the variety of metabelian groups. Then*

$$\mathcal{M}\mathcal{B}_2 \cap \mathcal{M}\mathcal{B}_3 \neq \mathcal{M}.$$

Indeed, there exists a finite group W such that $W^{(2)}$ and $W^{(3)}$ are both metabelian but W is of derived length 3.

The construction of W is rather involved and ad hoc, and does not have an obvious generalization. The smallest such W has order 1458.

This is yet another example of the chasm between nilpotency and solvability. Other properties that we lose when crossing from finitely generated nilpotent groups to finitely generated solvable groups include the following: residual finiteness, solvability of the word problem, polynomial growth, and finite presentability of the relatively free group.

As the free nilpotent group of class c is finitely presented, we know *a priori* that Corollary A1 will be true for fixed m and n if and only if it is provable in a very mechanical way, namely via a finite subpresentation of a canonical presentation for the free group of rank $c+1$ in the variety $\mathcal{N}_c\mathcal{B}_m \cap \mathcal{N}_c\mathcal{B}_n$ (in a way which we make precise in Section 2.5.1). Since such a finite presentation 'proving' the theorem for those m and n exists, it is natural to ask what such a presentation looks like: what is the minimum number of relators needed, does that number depend on m and n, and how must the specific relators change with m and n?

We analyse in detail the abelian case, where the answer to all of these questions is: surprisingly little.

Theorem C. *Let m and n be coprime. The following is a presentation of $\mathbb{Z} \times \mathbb{Z}$:*

$$\langle\, a, b \mid [a^m, b^m],\ [a^m, (ab)^m],\ [b^m, (ab)^m],\ [a^n, b^n],\ [a^n, (ab)^n],\ [b^n, (ab)^n] \,\rangle.$$

2.2 Basic notions

In this section we develop some basic tools which will be helpful, including aspects of the theory of group varieties. We also probe the definition of detectability in power subgroups: why specifically power subgroups, and what about the non-coprime case?

For the first question, there are easy examples showing that we cannot in general determine if a group law is satisfied just by examining two arbitrary subgroups, even if they are assumed to be normal and to generate the whole group: it is essential that we examine the characteristic, "verbally defined" power subgroups. For instance, the integral Heisenberg group $\langle\, x, y, z \mid [x, y] = z,\ [x, z] = [y, z] = 1 \,\rangle$ is the product of the two normal subgroups $\langle x, z\rangle$ and $\langle y, z\rangle$, which are both isomorphic to $\mathbb{Z}^2$, however the whole group is not abelian.

We now turn to the question of coprimality. For a property $\mathcal{P}$ of groups, we say a group G *has $\mathcal{P}$ coprime power subgroups* if there exist coprime m and n such that $G^{(m)}$ and $G^{(n)}$ both have the property $\mathcal{P}$. For example, using this terminology we can state the theorem of [Ven16] as: a finite group with abelian coprime power subgroups is abelian.

Notation. We write conjugation $g^h = h^{-1}gh$ and commutator $[g, h] = g^{-1}h^{-1}gh$.

An easily proved property of power subgroups (and verbal subgroups in general, see below for the definition) is the following:

Lemma 2.3. *Let $\varphi\colon G \twoheadrightarrow Q$ be a surjective group homomorphism and m an integer. Then $\varphi(G^{(m)}) = Q^{(m)}$ and $G^{(m)} \leq \varphi^{-1}(Q^{(m)})$.* □

Power subgroups pick up torsion elements:

Lemma 2.4. *Suppose that g has finite order r coprime to m. Then $g \in \langle g^m\rangle$.*

Proof. There exist integers x and y such that $xr + ym = 1$. Now

$$g = g^{xr+ym} = (g^r)^x (g^m)^y = (g^m)^y. \qquad \square$$

2.2.1 Varieties

We give a self-contained treatment of some basics from the theory of varieties of groups. For further details, the reader is referred to Hanna Neumann's classic book [Neu67].

Definition 2.5 (Variety of groups)**.** A *variety* of groups is the class of all groups satisfying each one of a (possibly infinite) set of laws.

Example 2.6. Corresponding to the examples of laws given above (immediately before Definition 2.1), we have the following examples of varieties:

- $\mathcal{A}$ – the variety of abelian groups;
- $\mathcal{B}_m$ – the 'Burnside' variety of groups of exponent m (or exponent dividing m, depending on the definition of exponent used);
- $\mathcal{M}$ – the variety of metabelian groups; *and*
- $\mathcal{N}_c$ – the variety of nilpotent groups of nilpotency class at most c.

Proposition 2.7. *A variety is closed under the operations of taking subgroups, quotients, and arbitrary Cartesian products.* □

In fact, *every* class of groups which is closed under these operations is a variety (see [Neu67, page 15.51]).

Definition 2.8 (Product variety)**.** Let $\mathcal{U}$ and $\mathcal{V}$ be varieties of groups. We define the *product variety* $\mathcal{UV}$ to be the class of groups which are an extension of a group from $\mathcal{U}$ by a group from $\mathcal{V}$. That is, $G \in \mathcal{UV}$ if there exists $N \lhd G$ such that $N \in \mathcal{U}$ and $G/N \in \mathcal{V}$. We define the product of two classes of groups similarly.

Example 2.9. Let $\mathcal{A}$ and $\mathcal{M}$ denote the varieties of abelian and metabelian groups, respectively. Then $\mathcal{M} = \mathcal{AA}$.

We check that the product variety is indeed a variety as follows. Let $\mathcal{V}(G) \leq G$ denote the *verbal subgroup* of G corresponding to $\mathcal{V}$, that is, the subgroup generated by the images of the defining laws of $\mathcal{V}$ under all maps $F_\infty \to G$. Thus $G \in \mathcal{V}$ if and only if the verbal subgroup $\mathcal{V}(G) = 1$. As defining laws for $\mathcal{UV}$ we take the images of the defining laws of $\mathcal{U}$ under all maps $F_\infty \to \mathcal{V}(F_\infty)$. Let G be a group and suppose $N \lhd G$, with $q\colon G \to G/N$ the natural homomorphism. Then $\mathcal{V}(G/N) = q(\mathcal{V}(G))$ (cf. Lemma 2.3), so the quotient G/N is in $\mathcal{V}$ if and only if

$\mathcal{V}(G) \leq N$. Thus $G \in \mathcal{UV}$ if and only if $N = \mathcal{V}(G)$ is in the variety $\mathcal{U}$. Every map $F_\infty \to \mathcal{V}(G)$ factors through some map $F_\infty \to \mathcal{V}(F_\infty)$, so we see that $\mathcal{V}(G) \in \mathcal{U}$ if and only if it satisfies every law which is the image of a defining law of $\mathcal{U}$ in $\mathcal{V}(F_\infty)$. We will explore this further in Section 2.5.

Proposition 2.10 ([Neu67, Theorem 21.51])**.** *The product of varieties of groups is associative.*

Thus the varieties form a monoid under product, and the unit is the variety $\mathbb{1}$ consisting of only the trivial group. We introduce a more restrictive notion of product for two classes of groups.

Definition 2.11 (Normal product class)**.** Let $\mathcal{C}$ and $\mathcal{D}$ be classes of groups. We define the *normal product* of $\mathcal{C}$ and $\mathcal{D}$, denoted $\mathcal{C} \odot \mathcal{D}$, to be the class of groups G with normal subgroups $C \in \mathcal{C}$ and $D \in \mathcal{D}$ such that $G = CD$.

In particular, $\mathcal{D} \odot \mathcal{C} = \mathcal{C} \odot \mathcal{D} \subseteq \mathcal{CD} \cap \mathcal{DC}$. This last inclusion can be proper, for example, $\mathcal{A} \odot \mathcal{A} \subset \mathcal{N}_2$ (by Theorem 2.20, due to Fitting), whereas $\mathcal{AA} = \mathcal{M}$.

Proposition 2.12. *Let G be a group, $\mathcal{V}$ a variety of groups, and m an integer. Recall that $\mathcal{B}_m$ denotes the Burnside variety of exponent m. The power subgroup $G^{(m)} \in \mathcal{V}$ if and only if $G \in \mathcal{VB}_m$.*

Proof. As in the proof that a product variety is a variety, we see that $G \in \mathcal{VB}_m$ if and only if the verbal subgroup $\mathcal{B}_m(G) \in \mathcal{V}$, and $\mathcal{B}_m(G) = G^{(m)}$. □

With this proposition in hand, we define a variety $\mathcal{V}$ to be *detectable in power subgroups* if, for all coprime m and n, we have $\mathcal{VB}_m \cap \mathcal{VB}_n = \mathcal{V}$ (the intersection of varieties is simply the intersection as classes of groups). In this chapter, we mostly encounter varieties that are finitely based, that is, that can be defined by finitely many laws, and thus by a single law (the concatenation of these laws written in distinct variables x_i); in this case, detectability of the variety is simply detectability of such a single defining law. It will be useful for us to understand how taking products of varieties interacts with taking intersection. Although we do not have left-distributivity, we do have some upper and lower bounds, as the next proposition indicates.

Proposition 2.13. *For all varieties $\mathcal{U}$, $\mathcal{V}$, $\mathcal{W}$ we have*

$$\mathcal{U}(\mathcal{V} \cap \mathcal{W}) \leq \mathcal{UV} \cap \mathcal{UW} \subseteq (\mathcal{U} \odot \mathcal{U})(\mathcal{V} \cap \mathcal{W}).$$

(We write $\subseteq$ as the last term is not a variety in general.)

Proof. The first inclusion is immediate, as $\mathcal{V} \cap \mathcal{W} \leq \mathcal{V}$ implies that $\mathcal{U}(\mathcal{V} \cap \mathcal{W})$ is contained in $\mathcal{U}\mathcal{V}$, and similarly in $\mathcal{U}\mathcal{W}$.

Now suppose that $G \in \mathcal{U}\mathcal{V} \cap \mathcal{U}\mathcal{W}$. This means G has normal subgroups $N_{\mathcal{V}}, N_{\mathcal{W}} \in \mathcal{U}$ such that $G/N_{\mathcal{V}} \in \mathcal{V}$ and $G/N_{\mathcal{W}} \in \mathcal{W}$. Let $N = N_{\mathcal{V}} N_{\mathcal{W}} \lhd G$. The group G/N will be a common quotient of $G/N_{\mathcal{V}}$ and $G/N_{\mathcal{W}}$, and thus in $\mathcal{V} \cap \mathcal{W}$, as varieties are closed under taking quotients. The kernel $N_{\mathcal{V}} N_{\mathcal{W}}$ is then a product of normal subgroups in $\mathcal{U}$, so it is in the class $\mathcal{U} \odot \mathcal{U}$. $\square$

Corollary 2.14. *Let m and n be coprime. Then for every variety $\mathcal{U}$,*

$$\mathcal{U}\mathcal{B}_m \cap \mathcal{U}\mathcal{B}_n \leq \mathcal{U} \odot \mathcal{U}.$$

Proof. Set $\mathcal{V} = \mathcal{B}_m$, $\mathcal{W} = \mathcal{B}_n$ in Proposition 2.13 and note $\mathcal{B}_m \cap \mathcal{B}_n = \mathcal{B}_{\gcd(m,n)} = \mathbb{1}$. $\square$

In contrast, we do have right-distributivity of product of varieties over intersection:

Proposition 2.15. *For all varieties $\mathcal{U}, \mathcal{V}, \mathcal{W}$ we have*

$$(\mathcal{U} \cap \mathcal{V})\mathcal{W} = \mathcal{U}\mathcal{W} \cap \mathcal{V}\mathcal{W}.$$

Proof. The inclusion "$\leq$" is immediate.

Suppose $G \in \mathcal{U}\mathcal{W} \cap \mathcal{V}\mathcal{W}$, with $N_{\mathcal{U}}, N_{\mathcal{V}} \lhd G$ such that $N_{\mathcal{U}}$ is in $\mathcal{U}$, $N_{\mathcal{V}}$ is in $\mathcal{V}$, and both quotients $G/N_{\mathcal{U}}, G/N_{\mathcal{V}}$ are in $\mathcal{W}$. We have a (generally non-surjective map)

$$G \to G/N_{\mathcal{U}} \times G/N_{\mathcal{V}}$$

with kernel $N_{\mathcal{U}} \cap N_{\mathcal{V}}$. That is, the kernel is in $\mathcal{U} \cap \mathcal{V}$, and the quotient is in $\mathcal{W}$, since a variety is closed under Cartesian product and subgroups. $\square$

Varieties are determined by their finitely generated groups:

Proposition 2.16. *Let $\mathcal{U}$ and $\mathcal{V}$ be varieties. Let $\mathcal{U}_f$ denote the subclass of groups $G \in \mathcal{U}$ such that G is finitely generated and define $\mathcal{V}_f$ similarly. Then $\mathcal{U} = \mathcal{V}$ if and only if $\mathcal{U}_f = \mathcal{V}_f$.*

Proof. Clearly $\mathcal{U} = \mathcal{V}$ implies $\mathcal{U}_f = \mathcal{V}_f$. Suppose $\mathcal{V}$ is *not* contained in $\mathcal{U}$, so there is a law $w \in F_\infty$ which is satisfied by all groups in $\mathcal{U}$, but there is some $G \in \mathcal{V}$ and $\varphi\colon F_\infty \to G$ with $\varphi(w) \neq 1$. The law w is a word on finitely many letters $x_1, \ldots, x_n$ in the basis for F_∞, and we can assume $\varphi(x_i) = 1$ for all $i > n$. The subgroup $G_0 \leq G$ generated by $\varphi(x_1), \ldots, \varphi(x_n)$ is an element of $\mathcal{V}_f$. We can consider φ as a map $F_\infty \to G_0$ and so G_0 does not satisfy the law w. Thus $\mathcal{V}_f$ is not contained in $\mathcal{U}$, so in particular $\mathcal{V}_f$ is not contained in $\mathcal{U}_f$. □

Recall a well-known fact about torsion groups, which we will apply several times.

Proposition 2.17 ([Rob96, page 5.4.11])**.** *A finitely generated solvable torsion group is finite.*

2.2.2 Coprimality

The notion of detectability of a law in power subgroups does not make sense in general if one allows m and n for which $\gcd(m, n) = d > 1$; this could only say something about $G^{(d)} \geq G^{(m)}, G^{(n)}$ and not the whole group G. For example, in a group G of exponent d the power subgroups are trivial and satisfy all laws $w \in F_\infty$, whereas G does not if it is non-trivial. A more extreme example is provided by the free Burnside groups of exponent d for large odd d, which are infinite by the celebrated work of Novikov and Adian, and thus are not even solvable (by Proposition 2.17).

A precise formulation of this idea is the following:

Proposition 2.18. *For every variety $\mathcal{V}$ and for all integers m and n, we have*

$$\mathcal{V}\mathcal{B}_m \cap \mathcal{V}\mathcal{B}_n \geq \mathcal{V}\mathcal{B}_{\gcd(m,n)}. \tag{$\star$}$$

Suppose further that $\mathcal{V}$ is detectable in power subgroups, that is, we have equality in ($\star$) for the case of coprime *m and n. Then we have equality in ($\star$) for* all *m and n.*

Proof. The inclusion ($\star$) is immediate, as both $\mathcal{B}_m$ and $\mathcal{B}_n$ contain $\mathcal{B}_{\gcd(m,n)}$.

Suppose now that $\mathcal{V}$ is detectable in power subgroups, and let $d = \gcd(m, n)$, $m' = m/d$, $n' = n/d$, so that m' and n' are coprime. We have $\mathcal{B}_m \leq \mathcal{B}_{m'}\mathcal{B}_d$ (in general this inclusion may be strict), and similarly for $\mathcal{B}_n$, and thus

$$\mathcal{V}\mathcal{B}_m \cap \mathcal{V}\mathcal{B}_n \leq \mathcal{V}\mathcal{B}_{m'}\mathcal{B}_d \cap \mathcal{V}\mathcal{B}_{n'}\mathcal{B}_d = (\mathcal{V}\mathcal{B}_{m'} \cap \mathcal{V}\mathcal{B}_{n'})\mathcal{B}_d$$

via right-distributivity of the product over intersection (Proposition 2.15), and implicitly using associativity of the variety product. By assumption of detectability, this last term is just $\mathcal{V}\mathcal{B}_d$. □

The reader is referred to [BO15] for more on the fascinating topic of products of Burnside varieties.

Remark 2.19 (More than two powers). The notion of detectability is unchanged if we replace the two powers m and n with powers $m_1, m_2, \ldots, m_k$ that are mutually (not necessarily pairwise) coprime, that is, $\gcd(m_1, m_2, \ldots, m_k) = 1$. This follows by an easy induction, which can be expressed conveniently using the characterization of Proposition 2.18.

2.3 Locally nilpotent varieties are detectable

The starting point for this section is a desire to generalize the result that commutativity is detectable in power subgroups to the nilpotent case. For instance, can power subgroups detect whether a group is nilpotent of class at most 2? We are carried quite a way towards our goal by Fitting's Theorem.

Theorem 2.20 (Fitting, [Rob96, page 5.2.8])**.** *Let M and N be normal nilpotent subgroups of a group G. If c and d are the nilpotency classes of M and N, then $L = MN$ is nilpotent of class at most $c + d$.*

However, this will only tell us, for instance, that if the power subgroups are nilpotent of class at most 2, then our group of interest is nilpotent of class at most 4. We first lay some foundations towards proving the general Theorem A, then see in Theorem 2.22 how we can reduce the bound of $2c$ to c, as in Corollary A1. By proving the general theorem, we will also be able to conclude that certain Engel laws are detectable.

Proposition 2.21. *Let m and n be coprime. Let $\mathcal{C}$ denote the class of nilpotent, locally nilpotent, solvable, or locally solvable groups. Then $\mathcal{C}\mathcal{B}_m \cap \mathcal{C}\mathcal{B}_n = \mathcal{C}$.*

Proof. In each of the first three cases, this is an application of Corollary 2.14 together with the corresponding standard result that the appropriate $\mathcal{C} \odot \mathcal{C}$ is equal to $\mathcal{C}$: Fitting's Theorem for the nilpotent case, the Hirsch–Plotkin Theorem [Rob96, page 12.1.2] for the locally nilpotent case, and the solvable case is elementary. (This in fact shows the result still holds after replacing $\mathcal{B}_m$ and $\mathcal{B}_n$ with two arbitrary varieties with trivial intersection.) However, a group which is the product of two normal locally solvable subgroups need not be locally solvable, as shown by P. Hall [Rob72, Theorem 8.19.1 (i)], so for the remaining case we exploit the power subgroup structure. This argument also allows us to conclude the locally nilpotent case from Fitting's theorem, without the need to invoke Hirsch–Plotkin.

Assume now that $G \in \mathcal{CB}_m \cap \mathcal{CB}_n$ is finitely generated, so that its quotient $G/G^{(m)}$ is finitely generated and of exponent m. By the second isomorphism theorem,

$$G/G^{(m)} \cong G^{(n)}/(G^{(m)} \cap G^{(n)})$$

and so since $G^{(n)}$ is locally solvable, its finitely generated quotient $G/G^{(m)}$ is solvable. Now $G/G^{(m)}$ is a finitely generated solvable torsion group, and thus finite (Proposition 2.17). Hence the subgroup $G^{(m)} \lhd G$ is of finite index, so it is finitely generated, and since groups in $\mathcal{U}$ are locally solvable, $G^{(m)}$ is in fact solvable. Similarly, $G^{(n)}$ is solvable. Thus $G = G^{(m)}G^{(n)}$ is solvable. □

Theorem 2.22. *Let G be a finitely generated nilpotent group and let m and n be coprime. If $G^{(m)}$ and $G^{(n)}$ both satisfy a law w, then G satisfies w.*

In other words, the variety generated by G is the intersection of the varieties generated by $G^{(m)}$ and $G^{(n)}$. (The variety generated by a group is the intersection all varieties containing it.)

Proof. Suppose for the sake of contradiction that there is a homomorphism $\varphi\colon F_\infty \to G$ with $\varphi(w) \neq 1$. Since G is finitely generated and nilpotent, it is residually finite [Rob96, page 5.4.17], so there is a map $q\colon G \twoheadrightarrow Q$ for some finite group Q such that $q(\varphi(w)) \neq 1$. As G is nilpotent, so is Q, and thus Q is the direct product of its Sylow subgroups [Rob96, page 5.2.4]. We compose q with a projection onto a Sylow subgroup in which $q(\varphi(w))$ has non-trivial image, to get $q_p\colon G \twoheadrightarrow Q_p$. Without loss of generality, p does not divide m so that $Q_p^{(m)} = Q_p$ (Lemma 2.4). This gives a contradiction, as $Q_p^{(m)} = q_p(G^{(m)})$ (Lemma 2.3), and $G^{(m)}$ satisfies the law w. □

Theorem A. *Let $\mathcal{V}$ be a locally nilpotent variety and let m and n be coprime. Then*

$$\mathcal{VB}_m \cap \mathcal{VB}_n = \mathcal{V}.$$

Proof. By Proposition 2.16, it suffices to consider finitely generated $G \in \mathcal{VB}_m \cap \mathcal{VB}_n$. Since $\mathcal{V}$ is locally nilpotent, Proposition 2.21 guarantees that G is locally nilpotent. As G is in fact finitely generated, we can now apply Theorem 2.22 to conclude that G satisfies every law which holds in both $G^{(m)}$ and $G^{(n)}$. Since $G^{(m)}$ and $G^{(n)}$ are in the variety $\mathcal{V}$, we conclude that $G \in \mathcal{V}$. □

The nilpotent groups of class at most c form the variety $\mathcal{N}_c$, so the following corollary is immediate.

Corollary A1. *Let m and n be coprime and let $c \geq 1$. Then a group G is nilpotent of class at most c if and only if $G^{(m)}$ and $G^{(n)}$ are both nilpotent of class at most c.* $\square$

This means that the precise nilpotency class of G is the maximum of the precise nilpotency classes of $G^{(m)}$ and $G^{(n)}$.

Corollary A2. *Let m and n be coprime and let $k \leq 4$. A group G is k-Engel if and only if $G^{(m)}$ and $G^{(n)}$ are both k-Engel.*

Proof. The variety of 4-Engel groups was shown to be locally nilpotent by Havas and Vaughan-Lee [HV05]. This also implies the (previously known) $k \leq 3$ cases, as it is clear from the definition $E_{k+1}(x,y) = [E_k(x,y), y]$ that a k-Engel group is also $(k+1)$-Engel. $\square$

Remark 2.23. Gruenberg proved that a locally solvable k-Engel group is locally nilpotent [Gru53], so the generality achieved in Proposition 2.21 would not help to establish detectability of a Engel law beyond the locally nilpotent case. For a survey on Engel groups, the reader is referred to [Tra11].

For the sake of completeness, and motivated by its importance in geometric group theory, we show that the class of virtually nilpotent groups (groups with a nilpotent subgroup of finite index) is detectable in power subgroups. We first prove a more general result, and then use the structure of subgroups as specifically power subgroups to argue that the precise 'virtual nilpotency class' is preserved.

Proposition 2.24. *Let G be a group with normal subgroups A and B such that $G = AB$. Suppose that A and B are virtually nilpotent. Then G is virtually nilpotent.*

Proof. To invoke Fitting's Theorem, we require nilpotent subgroups that are normal in G. Let A_0 be the normal subgroup of A which is nilpotent and of minimal finite index. Such an A_0 exists as A is virtually nilpotent, and it is unique by Fitting's Theorem, as the product of two finite index normal nilpotent subgroups of A is then a normal nilpotent subgroup of smaller index. (A_0 is the 'nilpotent radical' or 'Hirsch–Plotkin radical' of A.) Now A_0 is characteristic in A, and thus normal in G. We define B_0 similarly.

As A_0 and B_0 are both nilpotent and normal in G, their product A_0B_0 is nilpotent. Since A_0 and B_0 are finite index in A and B respectively, and normal in G, we conclude that A_0B_0 is finite index in $AB = G$. That is, G is virtually nilpotent. $\square$

Corollary 2.25. *Let G be a finitely generated group and let m and n be coprime. If $G^{(m)}$ and $G^{(n)}$ both have finite index subgroups which are nilpotent of class at most c, then so does G.*

Proof. By Proposition 2.24, G is virtually nilpotent. Thus $G/G^{(m)}$ has a finite index subgroup which is nilpotent, and moreover finitely generated and of exponent m, and hence finite (by Proposition 2.17). Now $G^{(m)}$ is finite index in G, so its finite index subgroups are finite index in G. $\square$

2.4 Derived length is not detectable

In this section we show by explicit example that one cannot extend the above results for the nilpotent case to the solvable case. Of course, Proposition 2.21 tells us that a group with solvable coprime power subgroups is itself solvable: the class of solvable groups is closed under extensions. The point is that we do not have precise control over derived length like we did for nilpotency class.

Theorem B. *Let $\mathcal{M}$ denote the variety of metabelian groups. Then*

$$\mathcal{M}\mathcal{B}_2 \cap \mathcal{M}\mathcal{B}_3 \neq \mathcal{M}.$$

Indeed, there exists a finite group W such that $W^{(2)}$ and $W^{(3)}$ are both metabelian but W is of derived length 3.

Proof. Let H_3 denote the mod-3 Heisenberg group, which is the non-abelian group of order 27 and exponent 3. It admits the presentation

$$H_3 = \langle\, x, y, z \mid x^3 = y^3 = 1,\, z = [x, y],\, [x, z] = [y, z] = 1 \,\rangle.$$

Write $\mathbb{Z}_n$ for the cyclic group of order n. The group W is constructed as

$$W := (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes_{\varphi} (H_3 \times \mathbb{Z}_2)$$

where, letting $\mathbb{Z}_2 = \langle t\rangle$, the action is defined by $\varphi\colon H_3 \to \mathrm{SL}_2(\mathbb{Z}_9)$ which maps

$$x \mapsto \begin{pmatrix} 1 & -1 \\ 3 & -2 \end{pmatrix}, \quad y \mapsto \begin{pmatrix} -2 & 0 \\ 0 & 4 \end{pmatrix}, \quad t \mapsto \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We check that this is a well-defined group action. (It is in fact faithful, however this is – while easily verified – unnecessary for the proof.) Let $X := \varphi(x)$, $Y := \varphi(y)$,

$T := \varphi(t)$, and $Z := [X, Y] = \varphi(z)$. As T is order 2 and central in $\mathrm{SL}_2(\mathbb{Z}_9)$, we only need to check the map $H_3 \to \mathrm{SL}_2(\mathbb{Z}_9)$. We see first that

$$X^2 = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix}, \quad Y^2 = \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix}$$

and thus $X^3 = Y^3 = 1$. As

$$XY = \begin{pmatrix} -2 & -4 \\ -6 & -8 \end{pmatrix} = \begin{pmatrix} -2 & -4 \\ 3 & 1 \end{pmatrix}$$

and

$$X^{-1}Y^{-1} = \begin{pmatrix} -2 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} 4 & 0 \\ 0 & -2 \end{pmatrix} = \begin{pmatrix} -8 & -2 \\ -12 & -2 \end{pmatrix} = \begin{pmatrix} 1 & -2 \\ -3 & -2 \end{pmatrix}$$

we see that

$$Z = \begin{pmatrix} 1 & -2 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} -2 & -4 \\ 3 & 1 \end{pmatrix} = \begin{pmatrix} -8 & -6 \\ 0 & 10 \end{pmatrix} = \begin{pmatrix} 1 & 3 \\ 0 & 1 \end{pmatrix}$$

so that Z has order 3, with $Z^{-1} = \begin{pmatrix} 1 & -3 \\ 0 & 1 \end{pmatrix}$.

We now determine the conjugation action of Z:

$$Z^{-1} \begin{pmatrix} a & b \\ c & d \end{pmatrix} Z = \begin{pmatrix} a - 3c & b - 3d \\ c & d \end{pmatrix} Z = \begin{pmatrix} a - 3c & b + 3(a - d) \\ c & d + 3c \end{pmatrix}$$

as $9c = 0$, so the matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is in the centralizer of Z precisely when both $a - d$ and c lie in $3\mathbb{Z}_9$. This is true for both X and Y, so we have $[X, Z] = [Y, Z] = 1$ as required, which completes the verification that φ is a well-defined group homomorphism (or in other words, X, Y and T generate a subgroup of $\mathrm{SL}_2(\mathbb{Z}_9)$ isomorphic to $H_3 \times \mathbb{Z}_2$, modulo the injectivity of φ which we will not verify here).

We claim that $W^{(2)} = (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes H_3$ and $W^{(3)} = (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes \mathbb{Z}_2$. The "$\leq$" inclusion is Lemma 2.3, and the other inclusion is not necessary for the proof and so is left to the curious reader (if it were not the case, it would only make the task at hand easier). The group $W^{(3)}$ is obviously metabelian, as it is exhibited as the semidirect product of one abelian group and another abelian group. On the other hand, $W^{(2)}$ will require the following basic computations.

Recall first the following:

Lemma 2.26. *Let $G = N \rtimes K$. Then the derived subgroup $G' = (N'[N, K]) \rtimes K'$.*

One can prove the lemma by verifying that every commutator in G lies in the subgroup generated by N', $[N, K]$ and K', and then noting that the action of K on N restricts to an action on $N'[N, K]$.

In the present case of $W^{(2)} = (\mathbb{Z}_9 \times \mathbb{Z}_9) \rtimes H_3$, since $N = \mathbb{Z}_9 \times \mathbb{Z}_9$ is abelian we simply have $[N, K] \rtimes K'$. The subgroup $[N, K]$ is generated by $(I - X)n$ and $(I - Y)n$ for $n \in \mathbb{Z}_9 \times \mathbb{Z}_9$. As

$$I - X = \begin{pmatrix} 0 & 1 \\ -3 & 3 \end{pmatrix}, \quad I - Y = \begin{pmatrix} 3 & 0 \\ 0 & -3 \end{pmatrix},$$

we see that $[N, K] = \mathbb{Z}_9 \times 3\mathbb{Z}_9 = \left\{ \begin{pmatrix} u \\ 3v \end{pmatrix} \,\middle|\, u, v \in \mathbb{Z}_9 \right\}$.

Now $H_3' = \langle z \rangle \cong \mathbb{Z}_3$, and we see that the set of invariants for Z is auspiciously none other than $\mathbb{Z}_9 \times 3\mathbb{Z}_9$. Thus $[N, K] \rtimes K'$ is abelian, so $W^{(2)}$ is metabelian.

On the other hand, for the negation action of $\mathbb{Z}_2$ on $N = \mathbb{Z}_9 \times \mathbb{Z}_9$ we have $[N, \mathbb{Z}_2] = 2N = N$, so we see that $W' = N \rtimes H_3'$, which has derived length 2, so W has derived length 3 as claimed. □

Remark 2.27. The group W constructed above has order $4374 = 2 \times 3^7$. It is also possible to construct a group W satisfying the requirements of the theorem as $(\mathbb{Z}_3 \times \mathbb{Z}_9) \rtimes (H_3 \times \mathbb{Z}_2)$, of order $1458 = 2 \times 3^6$ (the action is *not* simply a restriction or quotient of φ). An exhaustive search with `GAP` [GAP16] revealed that this is in fact the smallest non-metabelian group with metabelian coprime power subgroups. (There are two such groups of order 1458, and their ID pairs in the Small Groups Library are (1458, 1178) and (1458, 1192).)

Remark 2.28. We cannot extend this construction in an obvious way from the case of $p = 3$ to other primes. In particular, it appears to depend on the existence of a matrix of order p in $\mathrm{SL}_2(\mathbb{Z})$, which only has torsion elements of order at most 6.

Remark 2.29. One could ask for a finitely generated infinite group W that shows that being metabelian is not detectable in power subgroups. However, the failure will still be only up to finite index: such a group is solvable, and thus its power subgroups are of finite index (as used in the proof of Proposition 2.21).

2.5 Complexity analysis

By complexity analysis, we are *not* referring to analysis of algorithms and complexity classes such as `P` and `NP`, but the flavour is similar: we wish to quantify the complexity of detectability of given laws (in a sense we shall make precise), and understand the asymptotic behaviour of this complexity when the powers m and n vary.

We can formulate detectability of commutativity using an infinitely presented group having the appropriate universal property.

Proposition 2.30. *The detectability of commutativity in power subgroups is equivalent to the fact that, for all coprime m and n, the group $G_{m,n}$ defined by the infinite presentation*

$$G_{m,n} := \langle\, a, b \mid [u^m, v^m],\ [u^n, v^n] \quad \forall\, u, v \in F(a, b) \,\rangle$$

is isomorphic to $\mathbb{Z}^2$.

Proof. If $G_{m,n}$ is non-abelian then it is a counterexample. Suppose $G_{m,n}$ is abelian, and let H be a group with $H^{(m)}, H^{(n)}$ abelian. Then for every pair of elements $g, h \in H$ there is a homomorphism from $G_{m,n}$ to $\langle g, h\rangle \leq H$ defined by $a \mapsto g, b \mapsto h$, since all the relators have trivial image, and thus g and h commute. Therefore H is abelian. Moreover, as all relators in the presentation of $G_{m,n}$ are a consequence of the commutativity of the two generators, $G_{m,n}$ is in fact isomorphic to the free abelian group on 2 generators, $\mathbb{Z}^2$. $\square$

Thus for coprime m and n, the word $[a, b]$ is expressible in the free group as a product of conjugates of terms of the form $[u^m, v^m]$ and $[u^n, v^n]$. Such an expression gives a proof that $\mathcal{AB}_m \cap \mathcal{AB}_n = \mathcal{A}$. At this point, it is natural to ask how many different such terms are needed to encode such a proof. Before giving a very succinct choice of such terms, we phrase the set up in greater generality.

2.5.1 General framework

Definition 2.31. Let $\langle\, X \mid R \,\rangle$ be a group presentation. We call $\langle\, X \mid S \,\rangle$ a *subpresentation* of $\langle\, X \mid R \,\rangle$ if $S \subseteq R$. If moreover $\langle\!\langle S \rangle\!\rangle_{F(X)} = \langle\!\langle R \rangle\!\rangle_{F(X)}$, then we call $\langle\, X \mid S \,\rangle$ a *core* of $\langle\, X \mid R \,\rangle$.

A core not only defines an isomorphic group: there is a natural isomorphism induced by the identity map on the free group $F(X)$. We recall the following standard result.

Lemma 2.32. *Let $\langle\, X \mid R \,\rangle$ be a presentation of a group that admits a finite presentation, and assume that X is finite. Then $\langle\, X \mid R \,\rangle$ admits a finite core.*

Briefly, the proof is the following (see also [Mil04, Theorem 2.10]). Fix an isomorphism to the group defined by a finite presentation $\langle\, Y \mid S \,\rangle$, with the isomorphism and its inverse induced by $\varphi\colon F(X) \to F(Y)$ and $\psi\colon F(Y) \to F(X)$ respectively. These data also give an isomorphism for the group G defined by a subpresentation $\langle\, X \mid R' \,\rangle$ provided that $x =_G \psi(\varphi(x))$ for $x \in X$ and $\psi(s) =_G 1$ for $s \in S$. These

$|X| + |S|$ relations will each be a consequence of a finite subset R; the union of these gives us a finite choice for R'.

Let $\mathcal{V}$ be a finitely based variety, endowed with a chosen finite set $\mathcal{L}$ of defining laws. We only need finitely many variables for the laws in $\mathcal{L}$; suppose that each law is in $F_k \leq F_\infty$. Suppose further that the relatively free group $F_k/\mathcal{V}(F_k)$ is finitely presented (as an abstract group). For example, if $\mathcal{V}$ is nilpotent (or locally nilpotent) then this relatively free group is finitely generated and nilpotent, thus finitely presented. Now $\mathcal{VB}_m \cap \mathcal{VB}_n$ is equal to $\mathcal{V}$ if and only if its relatively free group of rank k is (naturally) isomorphic to $F_k/\mathcal{V}(F_k)$. That is, if and only if the infinite presentation $\mathcal{Q}$ defined by

$$\mathcal{Q}^{\mathcal{L}}_{k,m,n} = \langle\, x_1, \ldots, x_k \,|\, w(u_1, \ldots, u_k), w(v_1, \ldots, v_k) \text{ for } w \in \mathcal{L}, u_i \in F_k^{(m)}, v_i \in F_k^{(n)} \,\rangle$$

is a presentation for the finitely presented group $F_k/\mathcal{V}(F_k)$. If this is the case, then $\mathcal{Q}$ admits a finite core. Thus there is a partial algorithm that will decide if $\mathcal{Q}$ does indeed present $F_k/\mathcal{V}(F_k)$: enumerate larger and larger finite subpresentations of $\mathcal{Q}$ (that is, a filtration of the set of relators by finite sets) and attempt for longer and longer for each finite subpresentation to find a proof of isomorphism, proceeding in a diagonal fashion (we "diagonalize" the filtration and the isomorphism search). In general, the isomorphism problem is *partially* decidable (that is, there is an algorithm that will succeed in proving two input groups are isomorphic if they are isomorphic, but may fail otherwise), but our situation is easier, as we can fix the identity map on the generators (assuming our finite presentation for the relatively free group has k generators). So we only require a partial algorithm for the word problem: for instance, at the r-th attempt we can determine all words in F_k which are a product of conjugates of at most r relators or inverses of relators in the finite subpresentation, by words of length at most r, and freely reduce to see whether all the relators of the finite presentation for $F_k/\mathcal{V}(F_k)$ appear. Thus we have established the following:

Proposition 2.33. *Let $\mathcal{V}$ be a finitely based variety, and suppose that it admits a finite set of defining laws such that each law is on at most k variables. Suppose that the relatively free group $F_k/\mathcal{V}(F_k)$ is finitely presented as an abstract group. Then the set of coprime integers m and n for which*

$$\mathcal{VB}_m \cap \mathcal{VB}_n = \mathcal{V}$$

is a recursively enumerable set. That is, there is an algorithm which, given as input a pair m, n, will output YES and terminate in finite time if and only if the varieties are equal. □

However, we have only demonstrated the existence of such an algorithm: to actually implement the algorithm, we require additionally a finite presentation of $F_k/\mathcal{V}(F_k)$. For example, a presentation of the free 2-generator 4-Engel group was obtained by Nickel [Nic99, § 3.1]. To use Nickel's (polycyclic) presentation, where clearly only a_1 and a_2 are needed to generate the group, we use the obvious Tietze moves to remove the other generators; in general, given a finite presentation of $F_k/\mathcal{V}(F_k)$ on more than k generators, we could either enumerate presentations of the same group (in a blind search, via Tietze moves) until we construct one with k generators, or replace the partial algorithm for the word problem with a partial algorithm for the isomorphism problem (thereby deferring the difficulty).

The argument establishing Proposition 2.33 works in greater generality. In fact, all that we used about the varieties $\mathcal{B}_m$ and $\mathcal{B}_n$ is that they admit a basis which is a recursively enumerable subset of F_∞ (for example, a finite set). This is what implies that $F_k^{(m)} = \mathcal{B}_m(F_k)$ is recursively enumerable: for a general variety $\mathcal{U}$, each element of $\mathcal{U}(F_k)$ is a finite product of images of the defining laws, and each defining law has a recursively enumerable set of images in F_k. Under these conditions, we have a corresponding recursively enumerable presentation $\mathcal{Q}$ of the k-generated relatively free group in $\mathcal{VU} \cap \mathcal{VW}$. Thus, for $\mathcal{V}$ as in Proposition 2.33, there is a partial algorithm that takes as input the description of two recursively enumerable bases, for varieties $\mathcal{U}$ and $\mathcal{W}$, and will succeed in determining that $\mathcal{VU} \cap \mathcal{VW} = \mathcal{V}$ when this is the case. We do not need the assumption that we made in the "Burnside" case that the varieties $\mathcal{U}$ and $\mathcal{W}$ have trivial intersection, but if this were not the case we actually could have equality only if $\mathcal{V}$ were the variety of all groups [Neu67, page 23.32].

2.5.2 Complexity of the abelian case

For the abelian case, where $\mathcal{V} = \mathcal{A}$ defined by the law $[x, y]$, $k = 2$, and $F_2/\mathcal{A}(F_2) = F_2/[F_2, F_2] \cong \mathbb{Z}^2$, there is an extraordinarily short finite presentation of $\mathcal{Q}_{2,m,n}^{\{[x,y]\}}$.

Theorem C. *Let m and n be coprime. The following is a presentation of $\mathbb{Z} \times \mathbb{Z}$:*

$$\langle\, a, b \mid [a^m, b^m],\ [a^m, (ab)^m],\ [b^m, (ab)^m],\ [a^n, b^n],\ [a^n, (ab)^n],\ [b^n, (ab)^n] \,\rangle.$$

After proving this theorem, we became aware of another proof [MaSE] that groups with abelian power subgroups are abelian, from which one can extract a 2-generator 6-relator core of $\mathcal{Q}$ which defines $\mathbb{Z} \times \mathbb{Z}$, just as in Theorem C. However, our proof has the advantage of uniformity in the words from the verbal subgroup used, whereas in the other proof the length of the words grows quadratically with m and n.

The proof proceeds by first showing that the commutator $[a, b]$ is central; once we know this, the proof that it is trivial is very short.

Rather than prove that $G_{m,n}$ is nilpotent of class 2 directly, we instead prove the stronger result that the group Γ (defined below), an extension of $G_{m,n}$, is nilpotent of class 2. This group is moreover a common extension of all the $G_{m,n}$, so we see our introducing Γ as abstracting away m and n from the proof. (We will of course prove later that each $G_{m,n} \cong \mathbb{Z}^2$, and so technically $\mathbb{Z}^2$ itself is also a common extension *a posteriori*, but we are constructing a group which is *a priori* a common extension.)

Definition 2.34. Let the group Γ be defined by the presentation

$$\langle\, a, b, x, y, z \mid [a, x],\ [b, y],\ [ab, z],\ [x, y],\ [x, z],\ [y, z],\ [ax, by],\ [ax, abz],\ [by, abz]\,\rangle.$$

Lemma 2.35. *The group Γ is an extension of $G_{m,n}$, with the quotient map sending $a \mapsto a$ and $b \mapsto b$.*

Proof. As m and n are coprime, there exist integers p and q such that $pm - qn = 1$, that is, $pm = qn + 1$. Define a map $\Gamma \to G_{m,n}$ by $a \mapsto a$, $b \mapsto b$, $x \mapsto a^{qn}, y \mapsto b^{qn}$ and $z \mapsto (ab)^{qn}$. This is easily checked to be well defined, as every defining relator for Γ is mapped to a relator of the form $[u^k, v^l]$ for some u and v with $[u, v]$ a defining relator of $G_{m,n}$ and $k, l \in \mathbb{Z}$. $\square$

Proposition 2.36. *The subgroup $\langle a, b\rangle \leq \Gamma$ is nilpotent of class* 2*.*

Remark 2.37. The group Γ itself is nilpotent of class 2, with $[\Gamma, \Gamma] \cong \mathbb{Z}$. However, we confine ourselves here to proving Proposition 2.36, which is all that is required for Theorem C.

We prove Proposition 2.36 in a sequence of lemmas. It will be convenient to know that the symmetry in a and b of $G_{m,n}$ extends to Γ.

Lemma 2.38. *Let $\varphi : a \mapsto b \mapsto a, x \mapsto y \mapsto x, z \mapsto z^a$. Then φ is an automorphism of* Γ*.*

Proof. Since the above also defines an automorphism of the free group $F(a, b, x, y, z)$, it suffices to check that φ is a well-defined group homomorphism. To verify this we now show that the images of the relators are trivial, in the cases where this is not immediate. Note that since $[ab,z]=1$, we have $z^a = z^{b^{-1}}$.

$$\begin{aligned}
\varphi([ab,z]) &= [ba,z^a] = [ab,z]^a = 1\\
\varphi([x,z]) &= [y,z^a] = [y,z^{b^{-1}}] = [y,z]^{b^{-1}} = 1\\
\varphi([y,z]) &= [x,z^a] = [x,z]^a = 1\\
\varphi([ax,abz]) &= [by,baz^a] = [by,bza] = [by,zab]^{b^{-1}} = [by,abz]^{b^{-1}} = 1\\
\varphi([by,abz]) &= [ax,baz^a] = [ax,bza] = [ax,abz]^a = 1
\end{aligned}$$
□

Lemma 2.39. *The commutator $[ab,yx]=1$ in the group Γ.*

Proof. In light of Lemma 2.38, we can instead prove $[ba,xy]=1$ as follows:

$$\begin{array}{l|l}
axybabz & = abxz(ab)y\\
= (ax)(by)(abz) & = abx(ab)zy\\
= (abz)(ax)(by) & = abaxbyz\\
= abzxaby & = abaxybz.
\end{array}$$

After cancelling on the left and right, we have $xyba = baxy$. □

Lemma 2.40. *The commutator $[a,bzy]=1$ in the group Γ.*

Proof. We have

$$\begin{array}{l|l}
a(bzy) & = by(ab)zb^{-1}\\
= (abz)(by)b^{-1} & = byz(ab)b^{-1}\\
= (by)(abz)b^{-1} & = (bzy)a.
\end{array}$$

□

Lemma 2.41. *The commutator $[b,zax]=1$ in the group Γ.*

Proof. This follows from Lemma 2.40 by symmetry, as we have $\varphi([b,zax]) = [a,z^aby] = [a,z^{(ab)^{-1}a}by] = [a,z^{b^{-1}}by] = [a,bzy] = 1$. □

In the following computations, we will frequently use the basic fact that if $[g,hk]=1$ then $g^h = g^{k^{-1}}$.

Lemma 2.42. *The commutator $[b,z^{-1}x]=1$ in the group Γ.*

Proof. Note first that since abz and a both commute with ax, we have $[ax,bz]=1$ and thus $(ax)^b = (ax)^{z^{-1}}$. Now

$$\begin{array}{lll|l}
abxb^{-1} & = (ab)(xy)y^{-1}b^{-1} & & = (ax)^b\\
 & = (xy)(ab)y^{-1}b^{-1} & \text{Lemma 2.39} & = (ax)^{z^{-1}}\\
 & = yxay^{-1} & & = (ab)^{z^{-1}}(b^{-1}x)^{z^{-1}}\\
 & = b^{-1}(by)(ax)y^{-1} & & = abzb^{-1}xz^{-1}\\
 & = b^{-1}(ax)(by)y^{-1} & & = abzb^{-1}z^{-1}x.
\end{array}$$

Left-multiplying both sides by $z^{-1}b^{-1}a^{-1}$ gives $z^{-1}xb^{-1} = b^{-1}z^{-1}x$. □

Lemma 2.43. *The commutator* $[b,[a,z]] = 1$ *in the group* Γ.

Proof. By Lemma 2.41 we have $zax \in C_\Gamma(b)$, the centralizer of b, and by Lemma 2.42 we have $z^{-1}x \in C_\Gamma(b)$. The centralizer thus contains $z^{-1}xzax = xax$, and thus also

$$[xax, x^{-1}z] = [a,z]$$

since x is central in $\langle a,x,z\rangle$. □

We are now equipped to prove Proposition 2.36.

Proof of Proposition 2.36.

ab	$= (yx)ab(yx)^{-1}$	Lemma 2.39	$= a^{bz}b^{za}$	Lemmas 2.40 and 2.41
	$= yxay^{-1}bx^{-1}$		$= (a^{bz}b^{za})^{z^{-1}}$	LHS ab and z commute
	$= yay^{-1}xbx^{-1}$		$= a^b b^{zaz^{-1}}$	
	$= a^{y^{-1}}b^{x^{-1}}$		$= a^b b^a$	Lemma 2.43

Thus $b^a = (a^b)^{-1}ab = b^{-1}a^{-1}bab = b^{-1}b^a b$. Since b commutes with b^a, it commutes with $b^{-1}b^a = [b,a] = [a,b]^{-1}$. Applying φ, we see that also $[a,[a,b]] = 1$. Thus the subgroup $\langle a,b\rangle \leq \Gamma$ is nilpotent of class 2. □

Proof of Theorem C. As $[a,b]$ is central in $\langle a,b\rangle \leq \Gamma$ (Proposition 2.36) and Γ is an extension of $G_{m,n}$ (Lemma 2.35), it follows that $[a,b]$ is central in $G_{m,n}$. Thus $[a^m,b^m] = [a,b]^{m^2}$ and $[a^n,b^n] = [a,b]^{n^2}$. Since m and n are coprime, so are m^2 and n^2. Now coprime powers of $[a,b]$ are both trivial, so $[a,b] = 1$. □

Remark 2.44. It is not sufficient to take 5 of the 6 relators of $G_{m,n}$ (for coprime $m,n > 1$). Suppose that we omitted $[b^n,(ab)^n]$ (the other cases are analogous). A folklore theorem states that for all integers $p,q,r > 1$ one can find a finite group containing elements a and b such that a,b,ab have orders p,q,r respectively (for a proof, see [Mil13, Theorem 1.64]). Such a group for $(p,q,r) = (n,m,m)$ will be a quotient of the group defined by our truncated presentation, as all defining commutativity relators hold trivially by virtue of one term having trivial image. It cannot be abelian, as otherwise the order of ab would be mn.

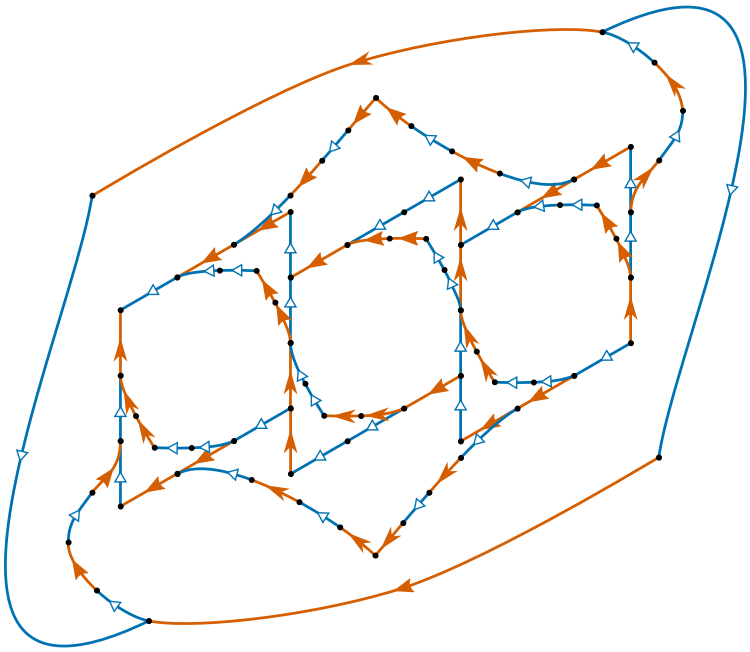


Figure 2.1: A van Kampen diagram proving that $[a, b] = 1$ in the group $\langle\, a, b \mid [a^2, b^2],\ [a^3, b^3],\ [(ab)^2, (ba)^2],\ [(ab)^3, (ba)^3] \,\rangle$

2.6 Open problems

While we have a surprisingly and uniformly small core (subpresentation) of $\mathcal{Q}$ in the abelian case, we still do not know what is the minimum number of relators needed.

Question 2.45. For which coprime m and n is there a 4-relator presentation for $\mathbb{Z}^2$ that encodes a proof that $\mathcal{AB}_m \cap \mathcal{AB}_n = \mathcal{A}$, that is, when does the infinite presentation $\mathcal{Q}_{2,m,n}^{\{[a,b]\}}$ have a 4-relator core? In particular, when is

$$\Delta_{m,n} = \langle\, a, b \mid [a^m, b^m],\ [a^n, b^n],\ [(ab)^m, (ba)^m],\ [(ab)^n, (ba)^n] \,\rangle$$

isomorphic to $\mathbb{Z}^2$?

The van Kampen diagram in Figure 2.1 proves that the question has a positive answer for $(m, n) = (2, 3)$. A van Kampen diagram *per se* is purely topological; the geometry of the drawing has been chosen such that corners generally delimit the 4 subwords u^{-1}, v^{-1}, u and v in a commutator $[u, v]$.

Computational experiments using `GAP` [GAP16] and `magma` [BCP97] provided some evidence for Question 2.45 having a positive answer. In particular, we have verified this for $m = 2$ and odd $n < 50$. However, we could not answer the question either way for $(m, n) = (3, 4)$. A necessary condition is for the four relators to generate the full *relation module*, that is, the abelianization of the kernel $F(a, b)'$ of the full presentation, as a $\mathbb{Z}[\mathbb{Z}^2]$-module. (This is actually a cyclic module.) This condition is met for $(m, n) = (3, 4)$; in fact, it is met for all $m \leq 80$ and $n = m + 1$.

Note that the argument of Remark 2.44 cannot be used to show that the group $\Delta_{m,n}$ has a non-abelian finite quotient: if we ask that $(ab)^m = 1$ and $(ba)^n = 1$, then since ab and ba are conjugate we in fact have $ab = 1$. Any possible non-abelian finite quotient is quite constrained; in particular, if the order of ab is coprime to either m or n then we immediately have $[ab, ba] = 1$, and similarly if a and b both have order coprime to m then $[a, b] = 1$. Meanwhile, we have determined computationally for the $(m, n) = (3, 4)$ case that if a and b have order dividing 24 then such a quotient is abelian. This computation was performed using Holt's package [Hol09] to show that the commutator subgroup of $\Delta_{3,4}/\langle\langle a^{24}, b^{24}\rangle\rangle$, an *a priori* finitely presented group, is trivial.

Question 2.46. Determine the analogous complexity for the nilpotency law

$$\nu_c = [[[\ldots[x_1, x_2], x_3], \ldots, x_c], x_{c+1}].$$

That is, how does the size of the smallest finite core of $\mathcal{Q}^{\{\nu_c\}}_{c+1,m,n}$ vary with c, m and n?

It seems that the following classification problem would require substantial progress.

Question 2.47. Which laws are detectable in power subgroups?

The difficulty is exemplified by the fact that the 4-Engel law is detectable, but it has been claimed that not all k-Engel laws imply the essential local nilpotency that we used. We thus ask in particular:

Question 2.48. For which k is the k-Engel law detectable in power subgroups.

To summarize our knowledge at this time, x^m is detectable in power subgroups, as is every locally nilpotent law (for example, the 4-Engel law $[x, y, y, y, y]$ or a nilpotency law such as $[[x_1, x_2], x_3]$). On the other hand, $[[x_1, x_2], [x_3, x_4]]$ is not detectable, and neither are some assorted laws for which detectability also fails in finite groups, such as $[[x^2, y^2]^3, y^3]$ and $[x^2, x^y]$.

Chapter 3

A census of small two-generator one-relator groups

3.1 Introduction

In this chapter we describe a census of all groups defined by a two-generator one-relator presentation with relator of length less than 10, which was completed in a combination of computation and manual pen-and-paper work. The census provides

- complete determination of isomorphisms classes, and
- determination of which groups have unbalanced Baumslag–Solitar subgroups and which groups are automatic.

The significance of the latter point will be explained in Section 3.1.1, which discusses the word problem for one-relator groups. A desire to develop our understanding of the word problem was the motivation for undertaking this census; we hope that the census will prove useful for other purposes. Section 3.1.2 describes some basic theory around the isomorphism problem for one-relator groups, which is applied in the census. In Section 3.2 we describe how the census was actually determined, and in Section 3.3 we present our findings on these (as we calculated) 134 groups. The full census is tabulated in Appendix A, listing all the isomorphism classes of groups with various properties we have determined.

Under the umbrella of studying the word problem, a specific objective of the census was to see if there exists a small counterexample to the following problem, which is a non-positively curved analogue of Gersten's infamous Question 1.1.

Problem 3.1 ([MUW11, Problem 1.5])**.** Is it true that every one-relator group with no unbalanced Baumslag–Solitar subgroup is automatic?

Recall that $\mathrm{BS}(m,n)$ is unbalanced if $|m| \neq |n|$. (As noted in Chapter 1, if we allow arbitrary groups of type F instead of just one-relator groups then the answer is no, by examples such as Example 1.4.) Against our intention, we establish that there is no such small counterexample, but we do however construct in Section 3.4.2 a counterexample to the closely related problem in which *CAT(0)* is substituted for *automatic*. For both of these purposes – giving positive and negative answers – we introduce a family of one-relator groups in Section 3.4. We classify when these groups are CAT(0), Theorem G, and show that some of the examples are automatic in Theorem 3.19. Using Bass–Serre theory, we deduce that this family gives counterexamples to the CAT(0) version of Problem 3.1 as Corollary G′. Finally, we discuss possible future work in Section 3.5 and give in Section 3.6 some technical proofs of isomorphism that are deferred so as not to interrupt the flow of reading.

3.1.1 The word problem

The theory of one-relator groups originated in the 1930s with Magnus's proof of the Freiheitssatz and thereby the solvability of the word problem.

Freiheitssatz ([Mag30]). *Let $G = \langle\, x_1, \dots, x_n \,|\, r \,\rangle$ be a one-relator group, where the relator r is cyclically (and freely) reduced. If L is a subset of $\{x_1, \dots, x_n\}$ that omits a generator that occurs in r, then L freely generates a free subgroup of G.*

Versions of the Freiheitssatz hold in limit groups and one-relator quotients of surface groups [HS09]. Magnus's proof used induction over a larger class of presentations, called *staggered presentations*. The modern formulation of Magnus's technique, pioneered by Moldavanskii, uses the Freiheitssatz to write each one-relator group as an HNN extension of a 'simpler' one-relator group over free subgroups. This gives a hierarchy, termed the Magnus–Moldavanskii hierarchy, for one-relator groups. The theory of normal forms for HNN extensions then allows us to solve the word problem, first achieved in different language in [Mag32].

Since the word problem is decidable for one-relator groups, one is lead to ask what its algorithmic complexity is. It is an open problem whether every one-relator group has polynomial time word problem. Myasnikov has even asked whether they all have word problem solvable in quadratic time [BMS+, (OR3)].

There are large classes of one-relator groups that are known to have quadratic time word problem. If a one-relator group has torsion (which is the case if and only if the relator is a proper power [KMS60]), then it is hyperbolic and has linear time word problem by the B. B. Newman Spelling Theorem [New68]. (Recent work of Wise implies more: these groups are virtually special and thus in particular residually finite.) A random one-relator group is small cancellation and thus hyperbolic as well.

More generally, many one-relator groups are automatic. Since a torsion-free one-relator group is of geometric dimension 2, Gersten's Theorem implies that if it has a Baumslag–Solitar subgroup then it cannot be automatic [Ger92]. It is an open problem whether this is the only obstruction; this is the substance of Problem 3.1.

Various related problems are also open: is every one-relator group with quadratic Dehn function automatic? Does the (hyperbolic) Gersten Question have a positive answer for one-relator groups?

Remark 3.2. Many two-generator one-relator groups are free-by-cyclic, and as we saw in Section 1.3.3 such groups have quadratic Dehn function but need not be automatic or CAT(0). However, the construction in Example 1.4 appears to depend essentially on having first Betti number at least 3, which is impossible for a two-generated group.

Remark 3.3. A one-relator group can be free-by-cyclic only if it is two-generated: A classifying space for a free-by-cyclic group is the mapping torus of a homotopy equivalence of a wedge of circles, which is 2-dimensional and of zero Euler characteristic. On the other hand, by Lyndon's theorem, a torsion-free group defined by a g-generator one-relator presentation has Euler characteristic equal to that of its presentation complex (a classifying space), which is $2 - g$.

A very important example for the word problem in one-relator groups is the "Baumslag–Gersten" group

$$G_{(1,2)} := \langle\, a, b \mid a^{a^b} = a^2 \,\rangle.$$

It was introduced in [Bau69], in which it was shown that all its finite quotients are cyclic; since it maps onto $\mathbb{Z}$, this implies $\widehat{G_{(1,2)}} \cong \widehat{\mathbb{Z}}$. Its Dehn function grows incredibly fast: it is

$$\underbrace{2^{2^{\cdot^{\cdot^{\cdot^{2}}}}}}_{\lfloor \log n \rfloor}.$$

This is conjecturally the largest Dehn function of all one-relator groups. In spite of this, it was shown to have word problem decidable in polynomial time using integer compression [MUW11]. This leads to another motivation for undertaking the census: to find new classes of one-relator groups with difficult word problem.

Up to length 9, we determine that there are no such examples: every group either has a fast word problem (via automaticity) or is in a known class of groups.

For our census, we restrict at present to two-generator groups. This simplifies the task substantially, but a natural objection is that this leaves out interesting groups (for instance, many examples of parafree groups are three-generator one-relator). However, in terms of the difficulty of the word problem, everything arises in the two-generator case, as the following lemma shows.

Lemma 3.4. *Every one-relator group embeds in a two-generator one-relator group.*

This is a special case of [HNN49, Theorem IV].

One family of automatic one-relator groups is the torus knot complements. There are many different ways we could conclude that these groups are automatic:

- working directly from the definition with automata;
- appealing to the fact that they are virtually $F_r \times \mathbb{Z}$;
- observing that they are central extensions of hyperbolic groups (automaticity by W. Neumann–Reeves [NR97a]);
- noting that they act geometrically on CAT(0) square complexes (automaticity by Niblo–Reeves [NR98]); *or*
- seeing them as fundamental groups of compact 3-manifolds without Nil or Solv geometry (automaticity by [ECH+92]).

3.1.2 Isomorphism of one-relator groups

The first part of our census is the determination of isomorphism types. Even if we were unable to carry this out completely, any groups which are identified as isomorphic potentially save us a lot of computation or human effort in determining automaticity (in fact, as we later note in Remark 3.14, a complete determination of isomorphism greatly assists us in establishing automaticity).

Let $I_k(n)$ denote the number of isomorphism types of k-generator one-relator groups with defining relators of length n. A remarkable theorem of I. Kapovich, Schupp and Shpilrain is the following.

Theorem 3.5 ([KSS06])**.** *There exist positive constants c_1 and c_2 depending on k but not on n such that*

$$\frac{c_1}{n}(2k-1)^n \leq I_k(n) \leq \frac{c_2}{n}(2k-1)^n.$$

So the number of isomorphism classes is within a multiplicative constant of the number of cyclic words of length n. This means that asymptotically we will not reduce the number of groups under consideration substantially by finding isomorphisms.

However, in practice, and in our small examples, the orders of magnitude are different. The 134 isomorphism types that we determine (up to length 9) is certainly at odds with the estimate of 2187 one gets from the bound of the theorem without the scaling constants c_1 and c_2; the payoff for whittling down the number of presentations to consider is well worth the effort.

The normal closure of a relator in the free group will not change if we take any conjugate, or in particular any cyclic permutation. Less naively, the isomorphism type of the one-relator group will not change if we were to swap a and b, or invert a generator. In general, applying any free group automorphism or inverting the relator will not change the isomorphism type.

Magnus had conjectured that this painted the entire picture as far as isomorphism for one-relator groups goes, and the conjecture appeared in the first edition of the classic text by Magnus, Karrass and Solitar.

Conjecture 3.6 (Magnus, [MKS66, p. 401])**.** *Let $G = \langle X \mid r \rangle$ and $H = \langle X \mid s \rangle$ be two one-relator groups. Then $G \cong H$ if and only if there exists $\varphi \in \operatorname{Aut}(F(X))$ such that $\varphi(r) = s$ or $\varphi(r) = s^{-1}$.*

This conjecture is known to be false; in the course of taking this census, we determine all 6 counterexamples up to length 9.

Table 3.1 records the number of one-relator presentations that we could have to consider at various stages of identifying isomorphic groups:

- taking reduced words in the free group;
- taking cyclic words (that is, conjugacy classes in F_2);
- taking cyclic words, up to the obvious length preserving automorphisms $\operatorname{Sym}^{\pm}(\{a, b\}) \cong \mathbb{Z}/2 \wr S_2$;
- taking $\operatorname{Aut}(F_2)$-orbits;
- taking $\operatorname{Aut}(F_2) \times \mathbb{Z}/2$-orbits, where $\operatorname{Aut}(F_2)$ acts naturally by automorphisms and $\mathbb{Z}/2$ acts by inversions (which commutes with applying automorphisms) to give a *permutation* action of the product on F_2 (N.B.: this action is not by group homomorphisms); *and*

equivalence	0	1	2	3	4	5	6	7	8	9	total
reduced words	1	4	12	36	108	324	972	2916	8748	26244	39365
cyclic words	1	4	8	12	26	52	132	316	836	2196	3583
cyclic, $\mathbb{Z}/2 \wr S_2$	1	1	2	2	6	7	20	40	114	275	468
$\mathrm{Aut}(F_2)$	1	1	1	1	3	4	10	16	43	101	181
$\mathrm{Aut}(F_2) \times \mathbb{Z}/2$	1	1	1	1	3	4	9	13	35	72	140
isomorphism	1	1	1	1	3	4	9	13	35	66	134

Table 3.1: Numbers of one-relator presentations, up to varying notions of equivalence, for given minimal relator length

- taking isomorphism classes.

Note that each equivalence class is counted precisely once, at the minimal relator length possible within that class.

Remark 3.7. The shortest word that is not automorphically equivalent to its inverse is $a^2b^2ab^{-1}$, and is the only such word of length 6 up to the obvious equivalence (that is, modulo the action of $\mathrm{Aut}(F_2) \times \mathbb{Z}/2$); this yields the first difference between the $\mathrm{Aut}(F_2)$ and $\mathrm{Aut}(F_2) \times \mathbb{Z}/2$ rows of the table. Coincidentally, the group $F_3 \rtimes \mathbb{Z} \cong \langle\, a, b \mid a^2b^2ab^{-1} \,\rangle$ (or, with a representative of the $\mathrm{Aut}(F_2)$-orbit of the inverse word $ba^{-1}b^{-2}a^{-2}$, the group $\langle\, a, b \mid a^2bab^{-2} \,\rangle$) was investigated by [BC07] and shown to have CAT(0) dimension 2 but CAT(-1) dimension 3.

The first counterexamples to Magnus's conjecture appeared in [Zie70] and [MP71]. Zieschang's example is a special case of the work of McCool–Pietrowski; the latter appeared in print later, although Zieschang does note that such examples by McCool–Pietrowski were to appear.

Note that in every one of our 6 pairs in Theorem E, at least one word has length 9; this is why the only difference between the last two rows of Table 3.1 is for relator length 9. A point of distinction between our examples of exceptional isomorphisms and those of previous authors is that we have examples where the minimal lengths in the two equivalence classes coincide. The previous proofs require different minimal lengths to conclude that the relators are in different equivalence classes; ours depends (for now, at least) on complete enumeration of the minimal length words in each class, and uses the following reasoning.

Automorphic equivalence is decidable by Whitehead's algorithm. There is a set of generators for $\mathrm{Aut}(F_n)$ called the *Whitehead automorphisms*. Whitehead proved that any word not of minimal length in its $\mathrm{Aut}(F_n)$-orbit can be shortened by applying one of these Whitehead automorphisms, and any two minimal length words in the orbit are related by a sequence of such automorphisms that preserve their length. This gives a greedy algorithm: reduce the input words to minimal length, then search over all minimal length words in the orbits (if the minimal lengths differ, we return *no* before the search). For this search, we simply do a graph search (for example, breadth first search [KT06, p. 79]) over the graph with vertices corresponding to minimal length elements in the automorphic orbits of our given words, and edges corresponding to Whitehead automorphisms taking one such word to another; the complexity of this search is linear in the number of vertices (after we fix n, so as to have a constant bound on degrees of vertices). Whitehead's algorithm takes *a priori* exponential time to determine if two elements are automorphically conjugate, because there are exponentially many words of a given length. However, in rank 2, Myasnikov and Shpilrain showed that the number of minimal length words in any orbit is polynomial in length, so the algorithm is polynomial time.

A large class of one-relator groups with Baumslag–Solitar subgroups is what we will call the "extended Baumslag–Solitar groups". (We cannot say *generalized Baumslag–Solitar groups* as this term refers to the finitely generated groups which act on a tree with all edge and vertex stabilizers isomorphic to $\mathbb{Z}$.)

Definition 3.8. Let m and n be non-zero integers, and let u and v be elements of the free group $F(X)$ such that $[u,v] \neq 1$. Then define the corresponding *extended Baumslag–Solitar group* as

$$\mathrm{EBS}(u,v,m,n) := \langle\, X \mid u^{-1}v^m u = v^n \,\rangle.$$

These groups were studied by Meskin [Mes72], who showed that they are not residually finite in the cases where the original Baumslag–Solitar groups are not residually finite, that is, if $|n| > |m| > 1$ (this is not an 'only if' result). Note that being an extended Baumslag–Solitar relator is preserved under the action of $\mathrm{Aut}(F_n)$.

Example 3.9. The non-linear residually finite group $\mathrm{DS} = \langle\, a,b \mid b^{-2}ab^2 = a^2 \,\rangle$ of Druţu–Sapir [DS05] is $\mathrm{EBS}(a, b^2, 1, 2)$. The Baumslag group $G_{(1,2)}$ is $\mathrm{EBS}(a, a^b, 1, 2)$.

3.2 Methods

Remark 3.10. As far as we know, there is no general partial algorithm to construct a census of all one-relator groups up to a certain input relator length n, including checking Problem 3.1 as we have done, even if we were to make various desirable but not very believable assumptions, for instance about profinite rigidity (which would at least give a solution to the isomorphism problem, see Proposition 1.15). To point the blame squarely, it is an open problem [MUW11, Problem 1.3] whether one can decide whether a one-relator group has a Baumslag–Solitar subgroup.

Our approach to extended Baumslag–Solitar groups is synthetic: we generate them for small choices of u, v, m, n. Alternatively, we could ask for each given relator whether it has the form $v^{-1}u^m v u^{-n}$. This was known to be decidable even before the solvability of general equations in free groups was proved by Makanin (see [Mes72, Theorem A] for a discussion of the specific case and [Mak82] for general equations).

3.3 Results of census

Theorem D. *Every two-generator one-relator group $\langle a, b \mid r \rangle$ of relator length $|r| \leq 9$ satisfies Problem 3.1.*

Theorem E. *The isomorphism classes of two-generator one-relator groups with relator length at most 9 are determined by automorphic orbit and inversion of the relator, except for the following 6 isomorphic pairs:*

$$\langle a, b \mid a^2b^2ab^{-1}a^{-2}b^{-1} \rangle \cong \langle c, d \mid c^3d^2cd^{-1}c^{-1}d \rangle \tag{3.1}$$
$$\langle a, b \mid a^3ba^2b^{-1} \rangle \cong \langle c, d \mid c^2dcd^{-1}cdcd^{-1} \rangle \tag{3.2}$$
$$\langle a, b \mid a^3ba^{-2}b^{-1} \rangle \cong \langle c, d \mid c^2dc^{-1}d^{-1}cdc^{-1}d^{-1} \rangle \tag{3.3}$$
$$\langle a, b \mid a^5b^2 \rangle \cong \langle c, d \mid c^3d^2c^2d^2 \rangle \tag{3.4}$$
$$\langle a, b \mid a^4bab^{-2} \rangle \cong \langle c, d \mid c^2dc^2d^{-1}cd^2 \rangle \tag{3.5}$$
$$\langle a, b \mid a^4ba^{-1}b^{-2} \rangle \cong \langle c, d \mid c^2dc^2d^{-1}c^{-1}d^2 \rangle \tag{3.6}$$

Concerning profinite rigidity, we prove:

Theorem F. *Let G and H be defined by two-generator one-relator presentations with relator length at most 9. Suppose that $\widehat{G} \cong \widehat{H}$. Then either $G \cong H$, or G and H are isomorphic to $\mathbb{Z}$ and $G_{1,2}$.*

We defer the proofs that these 6 pairs of groups are isomorphic to Section 3.6. The proof of Theorem D requires a by-hand verification of automaticity in two cases, which we perform in Section 3.4.3. We consider these two proofs, covered in Section 3.4.3 and 3.6, to be the only non-mechanical part of the otherwise computational proofs of the above theorems. That there are no fewer isomorphism classes than claimed is a consequence of Theorem F. After removing $G_{1,2}$ from consideration, the other 133 (isomorphism classes of) one-relator groups are distinguished from one another by the abelianizations of their finite index subgroups up to index 6; computing this only takes a matter of seconds.

Remark 3.11. The group $\mathrm{EBS}(a, a^b, 1, -2)$ has more finite quotients than the group $\mathrm{EBS}(a, a^b, 1, 2) \cong G_{(1,2)}$. For instance, it maps onto $S_3 \cong D_6$, sending a to rotation of order 3 and b to any reflection.

Further to the above theorems, which we consider the most important observations that can be extracted from the census, we have the following remarks.

Remark 3.12. All the groups considered in the census with Baumslag–Solitar subgroups are in fact themselves extended Baumslag–Solitar groups. There seems no reason to expect this to hold in greater generality.

Remark 3.13. This census includes only 7 possible candidates for a non-Hopfian automatic group (after observing that an ascending HNN extension of a free group is residually finite by [BS05]). It is an open question whether such a group exists. Wise claimed an example, but the proof of automaticity is acknowledged to be flawed; it is open whether that group in particular is automatic.

Remark 3.14. The time it takes `kbmag` to identify a group as automatic is highly sensitive to the presentation of the group given. After constructing the CAT(0) equilateral triangle complex on which the group $R(3, -2, 1, 1)$ of Section 3.4.3 acts, we ran `kbmag` on a presentation coming from the complex, and it succeeded in a matter of seconds. A naive computational search for an $\mathrm{Aut}(F_2)$-equivalent presentation on which `kbmag` runs in reasonable time was unsuccessful. However, for all but the two groups $R(3, -2, 1, 1)$ and $R(3, 2, 1, -1)$ which we examine in Section 3.4.3, we found equivalent relators (possibly longer) for which automaticity is quickly determined. In fact, the computational check of automaticity for the entire census can be carried out in around 2 minutes. Thus we consider our work to have produced an efficient certificate of automaticity in all relevant cases. (Of course, we could also store the entire automatic structure, but the trade-off between time and space seems unreasonable.)

As another example, the presentation $\langle\, a, b \mid a^2ba^{-1}b^{-2}a^{-1}b \,\rangle$ is determined to have an automatic structure in under a second, whereas $\langle\, a, b \mid aba^{-1}b^{-1}ab^{-1}a^{-1}b \,\rangle$ searched for over 8 hours before reaching the standard limits after which point the search is abandoned. (The first relator is sent to the second by the automorphism $a \mapsto a, b \mapsto a^{-1}b$.)

3.4 A family of one-relator groups

Definition 3.15. Let p, q, k and l be integers such that $|p|, |q| \geq 2$, $k \neq 0$ and $l \not\equiv 0 \mod p$. We define the group

$$R(p,q,k,l) := \langle\, x, y, t \mid x^p = y^q,\ (x^k)^t = x^l y \,\rangle.$$

Note that we can use a Tietze move to eliminate y, giving a two-generator one-relator presentation.

Remark 3.16. The group $R(p,q,k,l)$ is seen to be acylindrically hyperbolic by considering the action on its Bass–Serre tree $\mathcal{T}$ corresponding to the HNN extension with t as stable letter [MO15, Corollary 4.3].

Remark 3.17. We would eventually like to show that the groups $R(p,q,k,l)$ are not commensurable to one another. One tool for determining commensurability is however unavailable: a torsion-free 2-generator 1-relator group has all L^2-Betti numbers trivial [DL07].

3.4.1 CAT(0) classification

Theorem G. *The group $R(p,q,k,l)$ acts properly and cocompactly on a complete CAT(0) space if and only if $|k| > |l + \frac{p}{q}|$.*

Proof. For basic CAT(0) geometry, we frequently refer to [BH99, Chapter II.6]. Let X be a complete CAT(0) space and suppose that R acts properly and semi-simply on X (this is weaker than the assumption that the action is proper and cocompact [BH99, page II.6.10]). As R is torsion-free (since the relator in the one-relator presentation for R is not a proper power, or alternatively, since it is an HNN extension of a torsion-free group), every non-trivial element is hyperbolic and leaves invariant some axis on which it acts by translation.

Let $z = x^p = y^q$ be the generator of the centre of the torus knot complement group $T_{p,q} = \langle\, x, y \mid x^p = y^q \,\rangle$; this group is a subgroup of $R(p,q,k,l)$ in the obvious way, since $R(p,q,k,l)$ is the HNN extension $T_{p,q}*_{\mathbb{Z}}$. The set of points translated the minimal distance under the action of z is $\mathrm{Min}(z)$, which is isometric to a product $Y \times \mathbb{R}$ [BH99, II.6.8.(4), p. 231], where Y is the preimage of a point under the projection of $\mathrm{Min}(z)$ onto a single axis for z (and the product metric is the L^2 metric). Every $\gamma \in T_{p,q}$ commutes with z, so its action on X restricts to an action on $Y \times \mathbb{R}$ where it splits as (γ', γ'') for γ' an isometry of Y and γ'' a translation of $\mathbb{R}$ [BH99, II.6.8.(5), p. 231].

Up to scaling the metric on X, we can assume that z translates $\mathbb{R}$ by pq, so that x translates by q and y by p. Since the action of z on Y is trivial, x and y both act elliptically on Y, that is, they have a fixed point. Thus the translation length of x^k is $|kq|$.

In contrast, $x^l y$ must act with positive translation length on Y, as we now argue. Each element of $T_{p,q}$ achieves its translation length on the complete, convex and $T_{p,q}$-invariant subspace $Y \times \mathbb{R} \subseteq X$. The element $x^{p(lq+p)}(x^l y)^{-pq}$ translates $\mathbb{R}$ by 0, yet has infinite order (indeed, it has infinite order in the quotient $\mathbb{Z}/p * \mathbb{Z}/q$ since l is not a multiple of p), so it cannot fix a point in Y to achieve its positive translation length in $Y \times \mathbb{R}$. However, $x^{p(lq+p)}$ fixes Y pointwise, so $(x^l y)^{-pq}$ must have positive translation length on Y, and thus the same is true of $x^l y$. Putting these two directions together, $|x^l y| > |ql + p|$.

Conjugate group elements have the same translation length, so from the second relation we see that

$$|kq| = |x^k| = |x^l y| > |ql + p|$$

and thus

$$|k| > |l + \frac{p}{q}|.$$

Now suppose p, q, k, l satisfy the above inequality. The natural choice for a space on which to make $T_{p,q}$ act is the product of the Bass–Serre tree $\mathcal{T}_{p,q}$ for $\mathbb{Z}/p * \mathbb{Z}/q$ with a real line $\mathbb{R}$, where as in the min-set above we make x act on $\mathbb{R}$ by translation by q and y by p. This is a free cocompact action of $T_{p,q}$ on a CAT(0) space. We can scale the metrics in the $\mathcal{T}_{p,q}$ and $\mathbb{R}$ directions relative to each other to ensure that $|x^k| = |x^l y|$, since $|x^k|$ does not have any contribution from the $\mathcal{T}_{p,q}$ direction whereas $|x^l y|$ does. Once this condition is satisfied, we can glue the cylinder corresponding to the relation $(x^k)^t = x^l y$ to the quotient space [BH99, II.11.21, p. 358] and get a compact complex with fundamental group R whose universal cover is CAT(0). $\square$

3.4.2 Non-CAT(0) examples without distorted geometry

In this section we show that our family gives us counterexamples to the CAT(0) analogue of Problem 3.1, which is an alternative formulation of the Gersten conjecture for one-relator groups in non-positive curvature.

Proposition 3.18. *$R(p,q,k,l)$ has no unbalanced Baumslag–Solitar subgroup.*

Corollary G′. *Let k, l, p and q be integers as in Definition 3.15. Suppose that $|k| \leq |l + \frac{p}{q}|$. Then $R(p,q,k,l)$ is a one-relator group containing no unbalanced Baumslag–Solitar subgroup, and it does* not *act properly cocompactly on a complete CAT(0) space.*

As a concrete example, we could take $R(2,2,1,1)$.

Proof of Proposition 3.18. Consider the cyclic subgroups $A = \langle x^k \rangle$ and $B = \langle x^l y \rangle$ of the base group $G = \langle\, x, y \mid x^p = y^q \,\rangle$. Since l is not a multiple of p, the image $\overline{B}$ of B in the quotient $G/Z \cong \mathbb{Z}/p * \mathbb{Z}/q$ of G by the centre $Z = \langle x^p \rangle$ is of infinite order. On the other hand, the image $\overline{A}$ of A is of finite order. Since conjugation preserves order, no conjugate of $\overline{A}$ has non-trivial intersection with $\overline{B}$. As B maps injectively to $\overline{B}$, this lifts to give that $A^g \cap B = 1$ for all $g \in G$.

Proposition 6.3 of [But15] states that an HNN extension of a balanced group over infinite cyclic subgroups A and B, such that no conjugate of A intersects B, is again balanced. (Recall that balanced means that for infinite order g, if g^m and g^n are conjugate then $|m| = |n|$.) The base group G is CAT(0), so it is balanced (via the standard translation length argument), and thus R is balanced and in particular contains no unbalanced Baumslag–Solitar subgroup. □

3.4.3 Automaticity in the family

Theorem 3.19. *Let $R(p,q,k,l)$ be as in Definition 3.15. If $|k| = 2|l + \frac{p}{q}|$, then $R(p,q,k,l)$ is automatic.*

At present, we cannot prove any group $R(p,q,k,l)$ to be non-automatic.

The two relevant examples for the census are

$$\begin{aligned}
&\langle\, a, b \mid a^2bab^{-1}a^{-1}bab^{-1} \,\rangle \\
&\cong \langle\, a_0, a_1, t \mid a_0^2 a_1 a_0^{-1} a_1,\ a_0^t = a_1 \,\rangle \\
&\cong \langle\, a_0, a_1, t \mid a_0^3 (a_0^{-1} a_1)^2,\ a_0^t = a_0(a_0^{-1} a_1) \,\rangle \\
&\cong \langle\, x, y, t \mid x^3 = y^{-2},\ x^t = xy \,\rangle \\
&\cong R(3, -2, 1, 1)
\end{aligned}$$

and

$$
\begin{aligned}
&\langle\, a, b \mid a^2ba^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}\,\rangle \\
&\cong \langle\, a_0, a_1, t \mid a_0^2a_1^{-1}a_0^{-1}a_1^{-1},\ a_0^t = a_1\,\rangle \\
&\cong \langle\, a_0, a_1, t \mid a_0^3 = (a_0a_1)^2,\ a_0^t = a_0^{-1}(a_0a_1)\,\rangle \\
&\cong \langle\, x, y, t \mid x^3 = y^2,\ x^t = x^{-1}y\,\rangle \\
&\cong R(3, 2, 1, -1).
\end{aligned}
$$

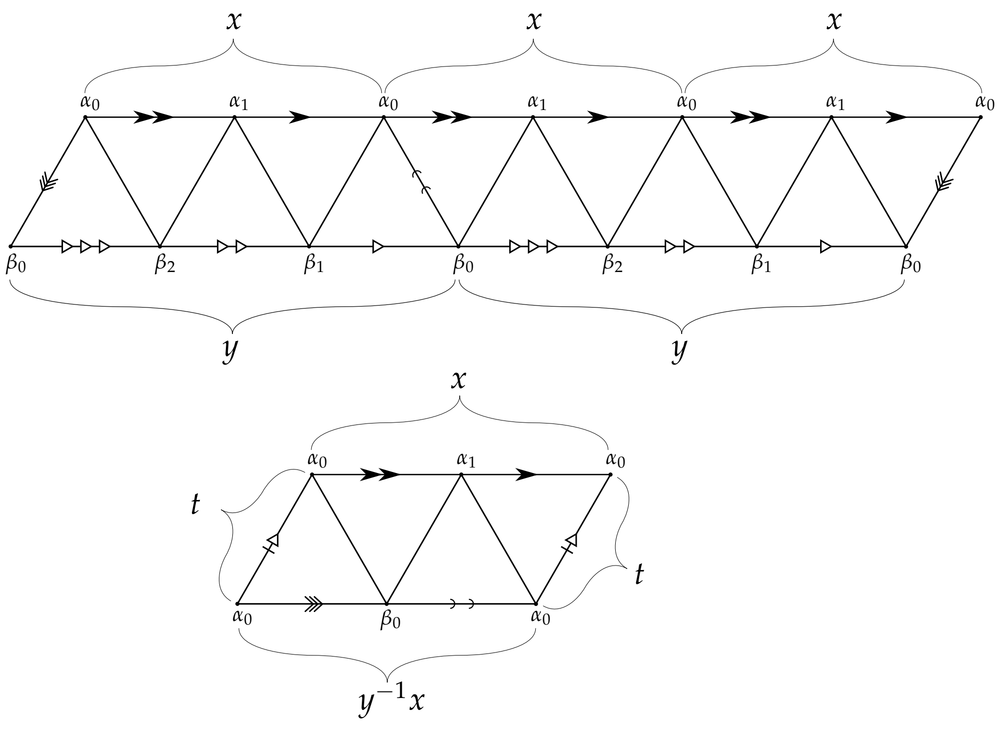


Figure 3.1: The non-positively curved equilateral triangle complex $X_{3,-2,1,1}$ with fundamental group $R(3,-2,1,1)$

The non-positively curved equilateral triangle complex with fundamental group $R(3,-2,1,1)$ is illustrated in Figure 3.1. The related space $\mathcal{T}_{3,-2} \times \mathbb{R}$ is shown in Figure 3.2.

(Note that throughout this chapter, we frequently make substitutions like $t = b^{-1}$, because the lexicographic order we apply to words when picking a representative relator for a given isomorphism class is at odds with our convention that conjugation acts on the right.)

Proof of Theorem 3.19. In this case, to arrange for x^k and $x^l y$ to have the same translation length on $\mathcal{T}_{p,q} \times \mathbb{R}$, as in the proof of Theorem G, we scale the two directions

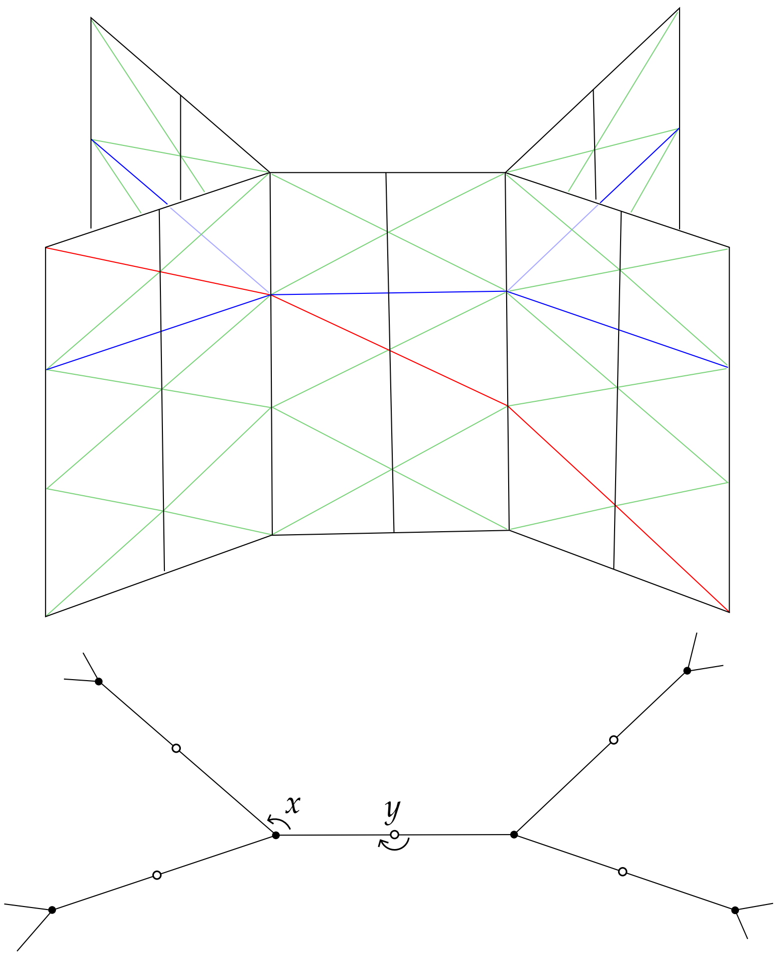


Figure 3.2: The CAT(0) equilateral triangle complex $\mathcal{T}_{3,-2} \times \mathbb{R}$ with the axis of xy indicated in red

so that all translations on $\mathbb{R}$ are integral and the translation length of $x^l y$ on $\mathcal{T}_{p,q}$ is $\sqrt{3}$ times its translation length on $\mathbb{R}$. It is then possible to tessellate $\mathcal{T}_{p,q} \times \mathbb{R}$ by equilateral triangles such that the action of $T_{p,q}$ preserves the triangulation, and the geodesics joining a vertex to its image under x^k or $x^l y$ are both in the 1-skeleton. Thus we can attach the cylinder corresponding to $(x^k)^t = x^l y$ to the quotient while preserving the structure as a non-positively curved equilateral triangle complex. One case of the Main Theorem of [GS90] is that the fundamental group of a non-positively curved equilateral triangle complex is automatic. $\square$

Remark 3.20. Another advantage of the equilateral triangle structure for the groups to which Theorem 3.19 applies is that it guarantees that all their finitely presented subgroups are also CAT(0) and automatic [BH99, II.5.30, p. 218]. This follows by a tower argument, as developed in the setting of 3-manifold topology by Papakyriakopoulos [Pap57] and adapted to combinatorial complexes by Howie [How81].

3.5 Future work

The obvious next steps are to extend the census to two-generator one-relator groups of longer relator length, and to one-relator groups with more than two generators. At present, this would require more than just asking more of existing software, as parts of the determination of the census require manual intervention. The salient example is the construction of the CAT(0) equilateral triangle complex on which the group R acts, since computational attempts to verify automaticity of this group failed. Although in theory the isomorphism problem is partially decidable, we were unable to achieve the 6 isomorphisms required up to length 9 purely computationally.

We do not expect that Baumslag–Solitar subgroups can be detected in greater generality by simply looking for EBS relators. Note that even in our modest census, the groups $\mathrm{BS}(2,3)$ and $\mathrm{BS}(2,-3)$ appear on the list of exceptional isomorphisms: they each have non-equivalent presentations. Attempting to identify Baumslag–Solitar subgroups algorithmically over larger and larger sets of examples will probably require a robust implementation of the Magnus–Moldavanskii hierarchy and a solution to the word problem, which would be no small task.

3.6 Proofs of exceptional isomorphisms

In this section we prove that the 6 exceptional pairs given in Theorem F are isomorphic.

Proof. (3.1) We first apply Magnus–Moldavanskii rewriting to express the groups as ascending HNN extensions of F_2. We replace b with t^{-1}, where t will be the stable letter. In both cases, we first obtain a three-generator one-relator presentation for the base group F_2 (with the relator primitive in F_3), which we then rewrite on two generators.

$$\begin{aligned}
&\langle\, a, t \mid a^2t^{-2}ata^{-2}t \,\rangle \\
&\cong \langle\, a_0, a_1, a_2, t \mid a_0^2a_2a_1^{-2},\ a_0^t = a_1, a_1^t = a_2 \,\rangle \\
&\cong \langle\, a_0, a_1, t \mid a_0^t = a_1,\ a_1^t = a_0^{-2}a_1^2 \,\rangle \\
&\cong \langle\, a, b, t \mid a^t = b,\ b^t = a^{-2}b^2 \,\rangle
\end{aligned}$$

after changing notation for the generators for convenience. On the other hand, $\langle\, c, d \mid c^3 d^2 c d^{-1} c^{-1} d \,\rangle$ does not come with a ready-made stable letter, so we apply a free group automorphism to arrange this, namely $c \mapsto c^{-2}d$ and $d \mapsto cd^{-1}c^2$, getting

$$\begin{aligned}
&(c^{-2}d)^3(cd^{-1}c^2)^2(c^{-2}d)(c^{-2}dc^{-1})(d^{-1}c^2)(cd^{-1}c^2)\\
&\sim dc^{-2}dcd^{-1}cdc^{-1}d^{-1}c
\end{aligned}$$

so, writing t for c, our group is isomorphic to

$$\begin{aligned}
&\langle\, d, t \mid dt^{-2}dtd^{-1}tdt^{-1}d^{-1}t \,\rangle\\
&\cong \langle\, d_0, d_1, d_2, t \mid d_0 d_2 d_1^{-1} d_0 d_1^{-1},\ d_0^t = d_1,\ d_1^t = d_2 \,\rangle\\
&\cong \langle\, d_0, d_1, t \mid d_0^t = d_1,\ d_1^t = d_0^{-1} d_1 d_0^{-1} d_1 \,\rangle\\
&\cong \langle\, c, d, t \mid c^t = d,\ d^t = (c^{-1}d)^2 \,\rangle
\end{aligned}$$

again changing notation for the generators for convenience.

Now

$$\begin{aligned}
&\langle\, a, b, t \mid a^t = b,\ b^t = a^{-2}b^2 \,\rangle\\
&\cong \langle\, a, b, t \mid a^t = b,\ (a^2)^t = b^2,\ b^t = a^{-2}b^2,\ (b^2)^t = (a^{-2}b^2)^2 \,\rangle\\
&\cong \langle\, a, b, c, d, t \mid a^t = b,\ (a^2)^t = b^2,\ b^t = a^{-2}b^2,\ (b^2)^t = (a^{-2}b^2)^2,\ c = a^2,\ d = b^2 \,\rangle\\
&\cong \langle\, a, b, c, d, t \mid a^t = b,\ c^t = d,\ b^t = c^{-1}d,\ d^t = (c^{-1}d)^2,\ c = a^2,\ d = b^2 \,\rangle\\
&\cong \langle\, a, b, c, d, t \mid a = b^{t^{-1}},\ c^t = d,\ b = (c^{-1}d)^{t^{-1}},\ d^t = (c^{-1}d)^2,\ c = a^2,\ d = b^2 \,\rangle\\
&\cong \langle\, a, c, d, t \mid a = (c^{-1}d)^{t^{-2}},\ c^t = d,\ d^t = (c^{-1}d)^2,\ c = a^2,\ d = ((c^{-1}d)^2)^{t^{-1}} \,\rangle\\
&\cong \langle\, c, d, t \mid c^t = d,\ d^t = (c^{-1}d)^2,\ c = ((c^{-1}d)^2)^{t^{-2}},\ d = ((c^{-1}d)^2)^{t^{-1}} \,\rangle\\
&\cong \langle\, c, d, t \mid c^t = d,\ d^t = (c^{-1}d)^2,\ c^{t^2} = (c^{-1}d)^2,\ d^t = (c^{-1}d)^2) \,\rangle\\
&\cong \langle\, c, d, t \mid c^t = d,\ d^t = (c^{-1}d)^2 \,\rangle
\end{aligned}$$

(3.2) The relator $c^2dcd^{-1}cdcd^{-1}$ is seen, as a consequence of our approach, to be minimal length in its $\mathrm{Aut}(F_2)$-orbit. However, once we apply Magnus–Moldavanskii rewriting, we have an easily reduced relator for the base group, namely a primitive word. This soon reveals the group to be isomorphic to $\mathrm{BS}(2, -3)$. We write t for d^{-1}.

$$\begin{aligned}
&\langle\, c, t \mid c^2t^{-1}ctct^{-1}ct \,\rangle\\
&\cong \langle\, c_0, c_1, t \mid c_0^2 c_1 c_0 c_1,\ c_0^t = c_1 \,\rangle
\end{aligned}$$

and since $c_0(c_0c_1)^2$ forms a basis of $F(c_0, c_1)$ together with c_0c_1, we see that the group $\langle\, c_0, c_1 \mid c_0^2c_1c_0c_1 \,\rangle$ is isomorphic to $\mathbb{Z} = \langle\, a \mid \,\rangle$ under $c_0 \mapsto a^2, c_1 \mapsto a^{-3}$. Thus our group is isomorphic to $\langle\, a, t \mid (a^2)^t = a^{-3} \,\rangle$.

(3.3) Likewise, we have (writing t for d^{-1})

$$\begin{aligned}&\langle\, c, t \mid c^2t^{-1}c^{-1}tct^{-1}c^{-1}t \,\rangle\\ &\cong \langle\, c_0, c_1, t \mid c_0^2c_1^{-1}c_0c_1^{-1},\ c_0^t = c_1 \,\rangle\end{aligned}$$

and $\langle\, c_0, c_1 \mid c_0^2c_1^{-1}c_0c_1^{-1} \,\rangle \cong \mathbb{Z} = \langle\, a \mid \,\rangle$ under $c_0 \mapsto a^2$, $c_1 \mapsto a^3$. So the group in question is isomorphic to $\langle\, a, t \mid (a^2)^t = a^3 \,\rangle$.

(3.4) This group appears as one of the examples of McCool–Pietrowski, but we include a proof here for completeness. For this, and the other remaining two groups, applying an automorphism to get a stable letter (of exponent sum zero) requires quite long manipulations, so we instead construct isomorphisms directly. Define $\varphi(a) = c^2d^2$, $\varphi(b) = d^{-1}$, and define $\psi(c) = a^{-2}$, $\psi(d) = b^{-1}$. Since $\varphi(a^2) = c^2d^2c^2d^2 = c^{-1}$, we have

$$\begin{aligned}\varphi(a^5b^2) &= \varphi(a^2)^2\varphi(a)\varphi(b)^2\\ &= c^{-2}(c^2d^2)d^{-2}\\ &= 1\end{aligned}$$

and as $\psi(d^2) = b^{-2} = a^5$, we have

$$\psi(c^3d^2c^2d^2) = a^{-6}a^5a^{-4}a^5 = 1,$$

so both maps are well-defined homomorphisms. Now $\varphi(\psi(c)) = \varphi(a^{-2}) = c$ and $\varphi(\psi(d)) = d$, so ψ is injective, and it is moreover surjective since $a = b^{-2}a^{-4} \in \langle a^2, b\rangle = \mathrm{im}(\psi)$. Thus ψ is an isomorphism.

(3.5) Again, we construct explicit isomorphisms. Define $\varphi(a) = c^{-2}d^{-1}$, $\varphi(b) = d^{-1}$, and define $\psi(c) = b^{-1}a^2b$, $\psi(d) = b^{-1}$. Since

$$\begin{aligned}\varphi(a^2) &= c^{-2}d^{-1}c^{-2}d^{-1}\\ &= c^{-2}d^{-1}c^{-2}(c^2dc^2d^{-1}cd^2)d^{-1} = d^{-1}cd\end{aligned}$$

we have

$$\varphi(a^4bab^{-2}) = (d^{-1}c^2d)d^{-1}(c^{-2}d^{-1})d^2 = 1.$$

Since

$$\psi(c^2) = b^{-1}a^4b = b^{-1}(a^4bab^{-2})^{-1}a^4b = ba^{-1},$$

we have

$$\psi(c^2dc^2d^{-1}cd^2) = (ba^{-1})b^{-1}(ba^{-1})b(b^{-1}a^2b)b^{-2} = 1.$$

Thus the maps are well defined. To verify that these are isomorphisms, it is easiest to show that they are mutually inverse, using the above computations of $\varphi(a^2)$ and $\psi(c^2)$. Clearly $\psi(\varphi(b)) = b$ and $\varphi(\psi(d)) = d$. Now

$$\psi(\varphi(a)) = \psi(c^{-2}d^{-1}) = \psi(c^2)^{-1}b = (ba^{-1})^{-1}b = a$$

and

$$\varphi(\psi(c)) = \varphi(b^{-1}a^2b) = d\varphi(a^2)d^{-1} = c.$$

(3.6) We construct explicit isomorphisms. Define $\varphi(a) = c^2d$, $\varphi(b) = d^{-1}$, and define $\psi(c) = (a^2)^{ba^{-1}}$, $\psi(d) = b^{-1}$. First, some useful computations:

$$\begin{aligned} \varphi(a^2) &= c^2dc^2d &&= (c^2dc^2d^{-1}c^{-1}d^2)^{-1}c^2dc^2d = c^{d^2} \\ \psi(c^2) &= (a^4)^{ba^{-1}} &&= (b^2ab^{-1})^{ba^{-1}} = ab. \end{aligned}$$

Thus

$$\begin{aligned} \varphi(a^4ba^{-1}b^{-2}) &= \varphi(a^2)^2d^{-1}(c^2d)^{-1}d^2 \\ &= (c^2)^{d^2}(c^{-2})^{d^2} = 1 \end{aligned}$$

and

$$\begin{aligned} \psi(c^2dc^2d^{-1}c^{-1}d^2) &= (ab)b^{-1}(ab)b(a^{-2})^{ba^{-1}}b^{-2} \\ &= a^2(a^{-2})^{ba^{-1}b^{-2}} \\ &= a^2(a^{-2})^{a^{-4}} = 1 \end{aligned}$$

so the maps are well defined. We have $\psi(\varphi(b)) = b$ and $\varphi(\psi(d)) = d$. Likewise

$$\begin{aligned} \psi(\varphi(a)) &= \psi(c^2d) &&= (ab)b^{-1} &&= a \\ \varphi(\psi(c)) &= \varphi((a^2)^{ba^{-1}}) &&= (c^{d^2})^{d^{-1}(c^2d)^{-1}} &&= (c^{d^2})^{d^{-2}c^{-2}} = c. \end{aligned}$$

□

Chapter 4

Finite p-groups of arbitrary negative deficiency

It is only fair to say that our knowledge of deficiency is quite deficient.

F. R. Beyl and J. Tappe, [BT82, p. 191]

4.1 Introduction

The *deficiency* of a group is the maximum over all presentations for that group of the number of generators minus the number of relators (some authors use the opposite sign convention). Every finite group has non-positive deficiency, since a group of deficiency at least 1 has infinite abelianization. For finite groups, most recent study of deficiency has focussed on finding deficiency zero presentations. The celebrated work of Golod and Shafarevich implies that a finite p-group of rank d has deficiency less than $-\frac{d^2}{4} + d$; this is one of many asymptotic results on deficiency of finite groups. On the other hand, the range of techniques for determining deficiencies of groups precisely is very limited. For example, the literature does not appear to contain a proof that all negative integers arise as deficiencies of finite groups. (The *infinite* case is easy: every integer is the deficiency of some $F_r * \mathbb{Z}^s$.) Another example of our lack of understanding of the fine structure of deficiency is an open problem in the Kourovka Notebook [MK14, 8.12(a)], due to D. L. Johnson and E. F. Robertson: Does there exist a finite p-group of rank 3 and deficiency zero for any $p \geq 5$? For rank $d \geq 4$ no such finite p-group exists, for any prime p, by Golod–Shafarevich.

In this chapter we prove the following theorem, which shows that indeed all negative integers arise as deficiencies of finite groups. The finite groups A_p, B_p and C_p – which are parameterized by a prime p – are introduced in Definition 4.6.

Theorem H. *Let p be a prime and $n \in \mathbb{N}$. Then there are natural numbers r, s and t such that the finite p-group $A_p^r \times B_p^s \times C_p^t$ has deficiency $-n$.*

A *Kähler group* is the fundamental group of a compact Kähler manifold. The class of Kähler groups includes all finite groups [Ser58], as well as surface groups and more generally the fundamental groups of complex projective varieties. Kotschick proved in [Kot12] that no Kähler group has even positive deficiency, and noted that this is the only constraint on positive deficiency for Kähler groups, as all odd positive integers arise as deficiencies of surface groups Σ_g. He then gave examples of Kähler groups of all negative deficiencies except for -5 and -7, with the suggestion that these should be achievable with finite groups (see Section 6 in [Kot12]).

Theorem H completes, with proof, the classification of deficiencies of Kähler groups, as suggested by Kotschick.

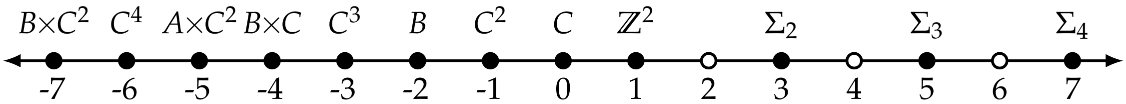


Figure 4.1: Deficiencies of Kähler groups

A search of the literature on deficiencies of finite groups suggests that one can extract examples as needed by Kotschick from the work of Sag and Wamsley, who claimed to have computed the deficiency of every group of order 2^n for $n \leq 6$ [SW73]. However, they did not publish proofs, and the article does contain a number of errors beyond the obvious misprints: some presentations are not *efficient* as claimed, and others do not define the groups they should. (To give one concrete example, the 252nd presentation of a group of order 64 is in fact a presentation of $\mathbb{Z}/4 \rtimes \mathbb{Z}/4$, for either commutator convention.) In future work, we will give a thorough analysis of their article and what is known about the deficiencies of small 2-groups.

The outline of our broad strategy to construct finite groups of arbitrary deficiency is as follows. We introduce in Section 4.2 the class $\mathcal{G}_p$ of efficient p-groups, in which we fully understand the deficiency of direct products (a quadratic polynomial). After finding enough basic examples in Section 4.3, we take repeated direct products in suitable combinations to obtain all negative integers as deficiencies; this analysis is the topic of Section 4.4. In analogy with results on the representability of positive integers by quadratic forms, such as the Conway–Schneeberger Fifteen Theorem [Bha00], there is good reason to expect such a strategy to succeed if we find enough basic building blocks in the class $\mathcal{G}_p$. The fact that the deficiency of our direct product is an inhomogeneous quadratic polynomial, rather than a quadratic form, makes the analysis easier, and we in fact only need the three basic examples A_p, B_p and C_p.

In the remainder of this chapter, we explore variations of our basic construction with building blocks having a different number of generators and relators (Section 4.5), enlarge our class of efficient groups to include infinite groups (Section 4.6), verify that Lustig's non-efficient group admits a minimal presentation (Section 4.7), and perform the Schur multiplier computation on which one of our variations depends (Section 4.8).

4.2 Controlling deficiency

For a group G, let $d(G)$ denote the minimal size of a generating set for G, which we call the *rank* of G. The homology groups $H_*(G)$ are implicitly taken with trivial $\mathbb{Z}$ coefficients, and tensor products are taken over $\mathbb{Z}$. In particular, the abelianization G/G' is isomorphic to $H_1(G)$. The deficiency of a group G is bounded above by

$$\operatorname{def}(G) \leq \operatorname{rk}(H_1(G)) - d(H_2(G)) \tag{$\star$}$$

where rk denotes the torsion-free rank of an abelian group: $\operatorname{rk}(G) = \operatorname{rk}_{\mathbb{Q}}(G \otimes_{\mathbb{Z}} \mathbb{Q})$. For a proof of this well-known inequality, the reader is referred to, for example, [BT07, Lemma 2] (N.B.: that article uses the opposite sign convention for deficiency). If a group achieves equality in ($\star$), then it is called *efficient*. The torsion-free rank of every finite group is zero, so the upper bound on deficiency of a finite group is simply minus the rank of the Schur multiplier $H_2(G)$.

Definition 4.1. A presentation realizing the deficiency of a group G is called *minimal* if it moreover has the minimal possible number of generators, namely the rank $d(G)$.

Note that we are asking more of a 'minimal' presentation than other authors; for example, [Gru79, § 4] only requires the number of generators of the group to be $d(G)$ with no requirements on the number of relators. A group can admit a minimal presentation without being efficient, as we show in Section 4.7. Indeed, it is an open problem whether every group admits a minimal presentation. Rapaport proved [Rap73] that this is the case for one-relator groups and nilpotent groups.

One class of finite groups where deficiency is reasonably understood is the class $\mathcal{G}_p$, for p a prime, as defined in [Joh70].

Definition 4.2. The class $\mathcal{G}_p$ denotes the finite p-groups G such that G is efficient and admits a minimal presentation.

In particular, the number of relators of such a presentation is simply $d(H_1(G)) + d(H_2(G))$, since every finite p-group G (indeed, every nilpotent group) satisfies $d(G) = d(H_1(G))$. (By Rapaport's theorem, one could remove the requirement of admitting a minimal presentation from the definition of $\mathcal{G}_p$.)

In fact, there is no known example of a non-efficient finite p-group.

Question 4.3 ([Man99, Question 18])**.** Is every finite p-group an element of $\mathcal{G}_p$?

The class $\mathcal{G}_p$ has been shown to be closed under various operations. For our purposes here, we only need closure under direct products as proved in [Joh70]. Since it is short and instructive, we include here a proof of this fact.

Lemma 4.4. *Let $G, H \in \mathcal{G}_p$. Then $G \times H \in \mathcal{G}_p$. Moreover, if minimal presentations are $G = \langle X \mid R \rangle$, $H = \langle Y \mid S \rangle$, then a minimal presentation for $G \times H$ is*

$$\langle X \sqcup Y \mid R \sqcup S \sqcup \{[x, y] \,:\, x \in X, y \in Y\} \rangle.$$

Proof. The above is a finite presentation of the finite p-group $G \times H$, and it has the required number of generators as $d(G \times H) = d(G) + d(H)$, since G and H are finite p-groups. It thus remains to prove that this is an efficient presentation, that is, that $G \times H$ has deficiency $-d(H_2(G \times H))$.

Recall that all the homology groups of a finite p-group are finite abelian p-groups. The special case of the Künneth formula proved by Schur states that

$$H_2(G \times H) \cong H_2(G) \oplus H_2(H) \oplus (H_1(G) \otimes H_1(H)).$$

As all four terms on the right-hand side are finite abelian p-groups, we see that

$$\begin{aligned} d(H_2(G \times H)) &= d(H_2(G)) + d(H_2(H)) + d(H_1(G)) \cdot d(H_1(H)) \\ &= |R| - |X| + |S| - |Y| + |X| \cdot |Y| \end{aligned}$$

and thus the presentation is efficient. □

Remark 4.5. As pointed out to us by Derek Holt, it is not difficult to attain finite groups of arbitrary (finite abelian) Schur multiplier. For every $n \geq 2$, if we choose q to be a prime power congruent to 1 modulo n, then $\mathrm{PSL}(n, q)$ is simple and has Schur multiplier cyclic of order n. (In the case $n = 2$ we require additionally $q \neq 3, 9$ and for $n = 3$ we need $q \neq 4$.) These groups are all perfect, so the Schur multiplier of a direct product of any of them will simply be the direct product of the Schur multipliers. Being perfect (and moreover simple) these groups are as far from the classes $\mathcal{G}_p$ as one could imagine for finite efficient groups, and there is no obvious way to construct efficient presentations of their direct products.

4.3 Building blocks

Fix a prime p. To construct our desired groups of arbitrary negative deficiency, we only need the following three groups from $\mathcal{G}_p$. Beyond being members of $\mathcal{G}_p$, the only relevant property of these basic examples is that the number of generators and relators in their minimal presentations are the pairs $(2, 2)$, $(2, 4)$ and $(1, 1)$.

Definition 4.6. Define groups by the presentations

$$\begin{aligned}
A_p &:= \langle\, a, b \mid a^p = b^p,\ a^b = a^{p+1} \,\rangle \\
B_p &:= \langle\, a, b \mid a^p,\ b^p,\ [[a, b], a],\ [[a, b], b] \,\rangle \\
C_p &:= \langle\, a \mid a^p \,\rangle
\end{aligned}$$

except when $p = 2$, where we define $B_2 := \langle\, a, b \mid a^4, b^4, (ab)^2, (a^{-1}b)^2 \,\rangle$.

Lemma 4.7. *A_p, B_p and C_p are all elements of $\mathcal{G}_p$ and the above presentations are minimal.*

Proof. First note that for each of the three presentations, the number of generators equals the rank of the abelianization, so minimality will follow once we establish that the presentations achieve the deficiency of their respective groups. Both A_p and $C_p \cong \mathbb{Z}/p$ are given by deficiency zero presentations, so to show they are elements of $\mathcal{G}_p$ it remains only to show that A_p is a finite p-group. (In fact, $A_2 \cong Q_8$ and $A_p \cong \mathbb{Z}/p^2 \rtimes \mathbb{Z}/p$ for odd p, but this is not needed for the proof.)

In A_p, the relation $a^b = a^{p+1}$ can be written as $[a, b] = a^p$, so since $a^p = b^p$ the commutator $[a, b]$ is central. Thus $[a, b]^p = [a^p, b] = [b^p, b] = 1$, so $a^{p^2} = (a^p)^p = [a, b]^p = 1$, and likewise $b^{p^2} = 1$, so A_p is nilpotent and generated by p-torsion, hence a finite p-group.

Table 8.1 in [Kar87] lists B_2, of order 16, as G_{15}, with Schur multiplier $(\mathbb{Z}/2)^2$, as proved by [Tah72]. For odd p, B_p is the mod-p Heisenberg group (of order p^3 and exponent p), with Schur multiplier $(\mathbb{Z}/p)^2$ [BT82, page 4.16]. Thus $B_p \in \mathcal{G}_p$. $\square$

4.4 The construction

We can now prove the main theorem, which we recall for the convenience of the reader.

Theorem H. *Let p be a prime and $n \in \mathbb{N}$. Then there are natural numbers r, s and t such that the finite p-group $A_p^r \times B_p^s \times C_p^t$ has deficiency $-n$.*

The proof is a combination of the building blocks from Section 4.3 together with some counting which we now abstract.

As noted before, if a group G is given by a presentation with m_1 generators and n_1 relators, and a group H is given by a presentation with m_2 generators and n_2 relators, then the standard presentation for $G \times H$ has $m_1 + m_2$ generators and $n_1 + n_2 + m_1 m_2$ relators. Define a binary operation $\diamond$ by

$$(m_1, n_1) \diamond (m_2, n_2) := (m_1 + m_2, n_1 + n_2 + m_1 m_2).$$

This operation is easily checked to be associative and commutative, so we get a commutative monoid $(\mathbb{N}^2, \diamond)$. In fact – although it does not greatly simplify the following proofs – this extends to give an alternate yet isomorphic group structure on $\mathbb{Z}^2$.

Lemma 4.8. *The map*

$$\begin{aligned} \varphi\colon (\mathbb{Z}^2, +) &\to (\mathbb{Z}^2, \diamond) \\ (x, y) &\mapsto (x, y + \tbinom{x}{2}) \end{aligned}$$

is an isomorphism of groups.

Note that we take the algebraic definition $\binom{x}{2} = \frac{x(x-1)}{2}$ for $x \in \mathbb{Z}$.

Proof. First note that φ is a bijection. Since

$$\begin{aligned} \binom{x_1 + x_2}{2} = \frac{(x_1 + x_2)(x_1 + x_2 - 1)}{2} &= \frac{x_1(x_1 - 1)}{2} + \frac{x_2(x_2 - 1))}{2} + x_1 x_2 \\ &= \tbinom{x_1}{2} + \tbinom{x_2}{2} + x_1 x_2 \end{aligned}$$

we see that

$$\begin{aligned}\varphi((x_1,y_1)) \diamond \varphi((x_2,y_2)) &= \left(x_1+x_2, (y_1+\tbinom{x_1}{2}) + (y_2+\tbinom{x_2}{2}) + x_1x_2\right)\\ &= \left(x_1+x_2, y_1+y_2+\tbinom{x_1+x_2}{2}\right)\\ &= \varphi((x_1+x_2, y_1+y_2))\end{aligned}$$

which completes the proof. □

To determine deficiency in direct products and powers, we use the following lemma. Let $(m,n)^{\diamond r}$ denote $\underbrace{(m,n)\diamond\cdots\diamond(m,n)}_{r\text{ times}}$.

Lemma 4.9.

$$(m_1,n_1)^{\diamond r_1}\diamond\cdots\diamond(m_k,n_k)^{\diamond r_k} = \left(\sum_i r_im_i, \sum_i r_in_i + \binom{\sum_i r_im_i}{2} - \sum_i r_i\binom{m_i}{2}\right)$$

Proof. After applying φ^{-1} to the left hand side we have

$$\sum_i r_i(m_i, n_i - \tbinom{m_i}{2}) = \left(\sum r_im_i, \sum_i r_in_i - \sum_i r_i\tbinom{m_i}{2}\right).$$

Applying φ gives the right hand side.

Alternatively, at least for $r_i, m_i, n_i \in \mathbb{N}$, we can count in the following way. Suppose that groups G_i are defined by m_i-generator n_i-relator presentations. The standard finite presentation for $G_1^{r_1}\times\cdots\times G_k^{r_k}$ has $\sum_i r_im_i$ generators. There are $\sum_i r_in_i$ relators coming from the relators in the direct factors. Additionally to this, commutativity relators are imposed for each unordered pair of generators *not* in the same direct factor. This is simply the number of unordered pairs amongst all generators, $\binom{\sum_i r_im_i}{2}$, minus the number of unordered pairs within the direct factors, which is $\sum_i r_i\binom{m_i}{2}$. □

We now show that our building blocks from Section 4.3 allow us to get arbitrary deficiency.

Definition 4.10. Define $\delta\colon \mathbb{Z}^2 \to \mathbb{Z}$ by $\delta(x,y) = x - y$.

Lemma 4.11. *For every $n \in \mathbb{N}$, there exist natural numbers r, s and t such that*

$$\delta\left((2,2)^{\diamond r}\diamond(2,4)^{\diamond s}\diamond(1,1)^{\diamond t}\right) = -n.$$

Proof. By Lemma 4.9, we see that the left hand side is

$$2r + 2s + t - \left[(2r + 4s + t) + \binom{2r + 2s + t}{2} - (r + s) \right]$$

so we need to find $r, s, t \in \mathbb{N}$ such that

$$\binom{2r + 2s + t}{2} + s - r = n.$$

Let m be the smallest positive integer such that $\binom{m}{2} + \lfloor \frac{m}{2} \rfloor \geq n$, and let $d := n - \binom{m}{2} \leq \lfloor \frac{m}{2} \rfloor$. By choice of m, we have $\binom{m-1}{2} + \lfloor \frac{m-1}{2} \rfloor \leq n$ (with equality only if $m = 1$). As $\lfloor \frac{m-1}{2} \rfloor + \lfloor \frac{m}{2} \rfloor = m - 1$ and $\binom{m-1}{2} + m - 1 = \binom{m}{2}$, we have $\binom{m-1}{2} + \lfloor \frac{m-1}{2} \rfloor = \binom{m}{2} - \lfloor \frac{m}{2} \rfloor$, so $d \geq -\lfloor \frac{m}{2} \rfloor$.

If $d \geq 0$, let $s := d$ and $r := 0$, and if $d < 0$ let $r := -d, s := 0$, so that in either case $s - r = d$ and $r + s \leq \lfloor \frac{m}{2} \rfloor$. Now we can let $t := m - 2r - 2s$, and thus $\binom{2r+2s+t}{2} + s - r = \binom{m}{2} + d = n$ as required. □

Proof of Theorem H. Combine Lemmas 4.7 and 4.11. □

Example 4.12. The group $A_p \times C_p^2$ has deficiency -5, and $B_p \times C_p^2$ has deficiency -7.

Remark 4.13. Since $H_1(G)$ and $H_2(G)$ are both elementary p-groups (that is, vector spaces over $\mathbb{Z}/p$) for odd p when G is any of our building blocks A_p, B_p and C_p, the construction actually gives, for each $n \in \mathbb{N}$, an efficient finite p-group whose Schur multiplier is $(\mathbb{Z}/p)^n$.

4.5 Variations on the construction

There are infinitely many alternatives for the groups A_p, B_p and C_p with the same numbers of generators and relators: $(2,2), (2,4), (1,1)$. For instance, instead of C_p we could take any cyclic p-group $\mathbb{Z}/p^m$. Our group $A_p = A_p^{1,1}$ also generalizes easily to

$$A_p^{m,n} := \langle\, a, b \mid a^{p^m} = b^{p^n} = [a, b] \,\rangle$$

and the proof of finiteness extends.

In fact, there are choices with *different* generator-relator pairs for which Theorem H holds (after proving in each case the appropriate version of Lemma 4.11). At least for $p = 2$ and $p = 3$ there are p-groups with a minimal 3-generator 3-relator presentation [JR79, § 4]. We can replace B_p by such a group, and find p-groups of arbitrary deficiency as direct products of deficiency zero groups with generator-relator pairs $(1,1), (2,2), (3,3)$.

Lemma 4.14. *For every $n \in \mathbb{N}$, there exist natural numbers r, s and t such that*

$$\delta\left((1,1)^{\diamond r} \diamond (2,2)^{\diamond s} \diamond (3,3)^{\diamond t}\right) = -n.$$

Proof. By Lemma 4.9, we see that *minus* the left hand side is

$$\binom{r+2s+3t}{2} - s - 3t.$$

We follow a similar strategy to the proof of Lemma 4.11. For a fixed $m = r + 2s + 3t$, we want to vary r, s and t such that $s + 3t$ takes the values $0, 1, \ldots, m-2$, and so thus $\binom{m}{2} - s - 3t$ takes all integer values from $\binom{m}{2} - (m-2) = \binom{m-1}{2} + 1$ to $\binom{m}{2}$ inclusive. We set $r := m - 2s - 3t$, so that the problem is reduced to solving $s + 3t = d \in \{0, 1, \ldots, m-2\}$ for $s, t \in \mathbb{N}$ subject to $2s + 3t \leq m$ (so that $r \geq 0$). Pick t to be as large as possible, that is, $t := \lfloor \frac{d}{3} \rfloor$, so that the remainder $s := d - 3t$ is in $\{0, 1, 2\}$. Since $s \leq 2$, we have $2s + 3t \leq s + 3t + 2 = d + 2 \leq m$ as required. □

Thus, we get something from (almost) nothing.

Corollary 4.15. *Every negative integer is the deficiency of a finite group which is the direct product of groups of deficiency zero.* □

For an alternative with pairs $(1,1), (2,4), (2,5)$, we could replace A_p with $\mathrm{J}_p := \langle\, a, b \mid a^{p^2},\, b^{p^2},\, [[a,b],a],\, [[a,b],b],\, [a,b]^p \,\rangle$ which is order p^5 and has Schur multiplier $(\mathbb{Z}/p)^3$. Proving that J_p has these properties, however, requires work, and is the subject of Section 4.8.

Lemma 4.16. *For every $n \in \mathbb{N}$, there exist natural numbers r, s and t such that*

$$\delta\left((1,1)^{\diamond r} \diamond (2,4)^{\diamond s} \diamond (2,5)^{\diamond t}\right) = -n.$$

Proof. This time Lemma 4.9 says that we need to solve

$$\binom{r+2s+2t}{2} + s + 2t = n.$$

Let m be the largest integer such that $\binom{m}{2} \leq n$, and fix $r := m - 2s - 2t$. Thus we require $s + 2t = n - \binom{m}{2} \leq m - 1$ subject to $2s + 2t \leq m$. Pick $t := \lfloor \frac{1}{2}(n - \binom{m}{2}) \rfloor$ so that $0 \leq s \leq 1$. Now $2s + 2t \leq s + 2t + 1 \leq m$. □

The problem of generating arbitrary negative deficiency, phrased in $(\mathbb{Z}^2, \diamond)$, is that we need to show that our finitely-generated submonoid of $(\mathbb{N}^2, \diamond)$ has, for each $n \in \mathbb{N}$, an element (k, l) with $k - l = -n$. After translating this problem

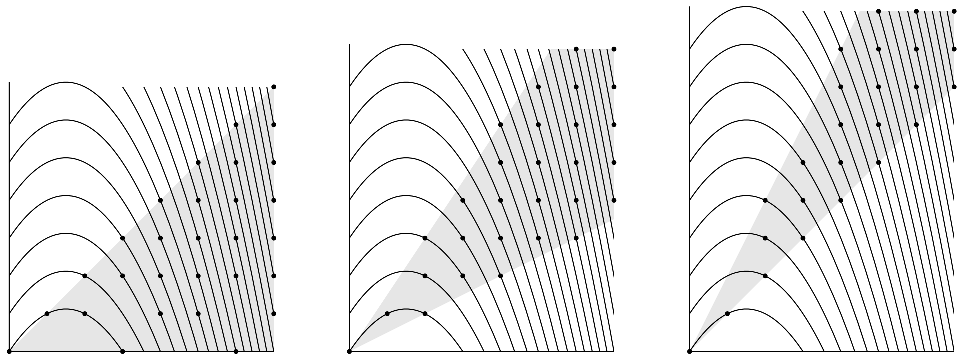


Figure 4.2: Generating arbitrary deficiency with $\{(1,1),(2,1),(3,0)\}$, $\{(1,1),(2,1),(2,3)\}$, and $\{(1,1),(2,3),(2,4)\}$ in $(\mathbb{Z}^2,+)$

from $(\mathbb{Z}^2,\diamond)$ to $(\mathbb{Z}^2,+)$, we need to show that, for each n, there is a non-negative $\mathbb{Z}$-linear combination of our generators lying on the parabola $y = n - \frac{x(x-1)}{2} + x$. This is plotted in Figure 4.2. Translating the generators for our 3 examples, namely Lemmas 4.14, 4.11, and 4.16, we have

$$\begin{aligned}(\mathbb{Z}^2,\diamond) &\to (\mathbb{Z}^2,+)\\ \{(1,1),(2,2),(3,3)\} &\mapsto \{(1,1),(2,1),(3,0)\}\\ \{(1,1),(2,2),(2,4)\} &\mapsto \{(1,1),(2,1),(2,3)\}\\ \{(1,1),(2,4),(2,5)\} &\mapsto \{(1,1),(2,3),(2,4)\}\end{aligned}$$

respectively. The cone in grey indicates the non-negative $\mathbb{R}$-linear combinations of the generators. Note that the lower and upper limits of the cone have gradients 0 and 1, $\frac{1}{2}$ and $\frac{3}{2}$, and 1 and 2 respectively; in each case the difference in 1. This is in fact a necessary condition: if the cone were narrower, then since

$$\left[-\frac{(x+1)x}{2} + (x+1)\right] - \left[-\frac{x(x-1)}{2} + x\right] = 1 - x,$$

for large enough n, the parabola $y = n - \frac{x(x-1)}{2} + x$ will be above the cone for some x_0 and below it for $x_0 + 1$, and thus will not intersect it at any lattice point in $\mathbb{Z}^2$.

On the other hand, provided that the generators are such that the *subgroup* of $\mathbb{Z}^2$ that they generate is all of $\mathbb{Z}^2$, then the submonoid will include all lattice points at least some *constant* amount inside the cone, by Lemma 4.18 below. So provided the difference in gradients is *strictly greater* than 1, such a generating set will at least give all but finitely many negative deficiencies.

At the critical value of 1, it is possible to generate all or all but finitely many deficiencies, and it is also possible that infinitely many values are missed. For

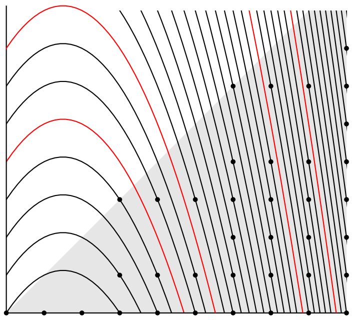


Figure 4.3: The set $\{(1,0),(3,1),(3,3)\}$ in $(\mathbb{N}^2,+)$ fails to generate

completeness we give here an example, which is illustrated in Figure 4.3. The red parabolas indicate values of deficiency that are not attained.

Example 4.17. The submonoid of $(\mathbb{Z}^2,+)$ generated by $(1,0)$, $(3,1)$ and $(3,3)$ does not contain any points on $y = n - \frac{x(x-1)}{2} + x$ for $n = 9\binom{k}{2} - 2$, where k is any positive integer.

At $x = 3k$, the quadratic function takes the value $(9\binom{k}{2} - 2) - \frac{x(x-3)}{2} = -2$, and is certainly not a positive $\mathbb{Z}$-linear combination of the generators. At $x = 3k-1$, it takes the value $3k-4$. In order to lie on the line $x - y = 3$, since the values of $x - y$ for the generators are $1, 2$ and 0, the contribution of $(1,0)$ and $(3,1)$ must be $3 \cdot (1,0) + 0 \cdot (3,1)$ or $1 \cdot (1,0) + 1 \cdot (3,1)$. These however force the x-value to be 0 or 1 modulo 3, so we cannot then add an integer multiple of $(3,3)$ to hit $(3k-1, 3k-4)$.

We now state a lemma that we expect should be familiar or at least unsurprising to experts in the geometry of numbers.

Lemma 4.18. *Let $S = \{(x_1,y_1),\dots,(x_k,y_k)\}$ be a set of points in $\mathbb{Z}^2$ with $x_i > 0$, and assume without loss of generality that $\frac{y_i}{x_i}$ is minimal for $i = 1$ and maximal for $i = k$. Suppose that S generates $\mathbb{Z}^2$ as a group. Then there is a constant C such that every lattice point in the sector bounded by the rays $y = \frac{y_1}{x_1}x$ and $y = \frac{y_k}{x_k}x$ (for $x \in \mathbb{R}^+$) at distance at least C from the boundary is in the submonoid generated by S.*

The following proof was suggested by Jakub Konieczny. It is illustrated in Figure 4.4.

Proof. Consider the subgroup L of $\mathbb{Z}^2$ generated by (x_1,y_1) and (x_k,y_k). Let (a_j,b_j) be the elements of $\mathbb{Z}^2$ of the form $\alpha(x_1,y_1) + \beta(x_k,y_k)$ for $0 \le \alpha,\beta < 1$, so that they are representatives for the finitely many cosets of L. Each is expressible as

$$(a_j,b_j) = \sum_{i=1}^{k} \alpha_{i,j}(x_i,y_i)$$

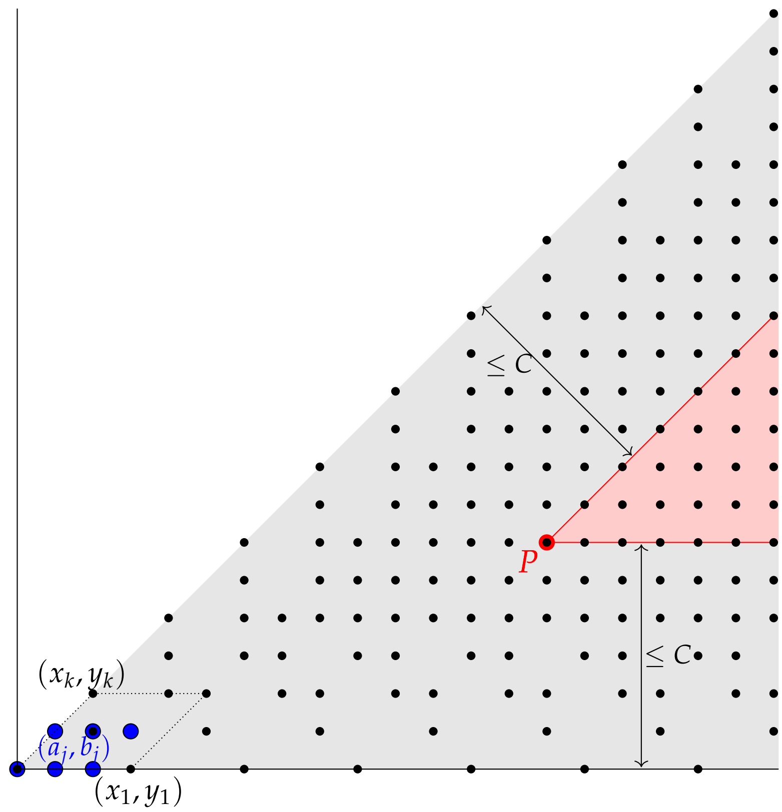


Figure 4.4: Illustration of Lemma 4.18 for $S = ((3,0),(2,1),(2,2))$

for $\alpha_{i,j} \in \mathbb{Z}$. Let $N = \max_{i,j}(-\alpha_{i,j})$. Let $P = N\sum_{i=1}^{k}(x_i, y_i)$. Then we can take C to be the greater of the Euclidean distances from the two boundary lines of the sector to P. This is because then any point at distance at least C from the boundary will be expressible as $P + Q$ for some Q in the sector, and we can write $Q = m(x_1, y_1) + n(x_k, y_k) + (a_j, b_j)$ for some integers $m, n \geq 0$ and coset representative (a_j, b_j) (for m and n we can take the floor of the $\mathbb{Q}$-linear coordinates). By construction of P, the negative contribution of any point (x_i, y_i) in writing (a_j, b_j) as a $\mathbb{Z}$-linear combination of S will be offset, and thus our point is in the submonoid generated by S. $\square$

4.6 Adding in infinite groups

The closure under direct product of $\mathcal{G}_p$, and the corresponding determination of deficiency, holds in a larger class $\mathcal{H}_p$ of efficient groups which we now introduce. This class is of interest both for its own sake, and also because it includes infinite Kähler groups, such as the surface group Σ_g.

We write $b^k(G, \mathbb{F}_p) := \dim_{\mathbb{F}_p} H^k(G, \mathbb{F}_p)$. As above, all homology and cohomology is taken with trivial coefficients, which are implicitly $\mathbb{Z}$ if not specified.

Definition 4.19. $\mathcal{H}_p$ is the class of finitely presented groups G such that G admits a presentation on $b^1(G, \mathbb{F}_p)$ generators and $b^2(G, \mathbb{F}_p)$ relators.

The reason why we should consider this class is that there is a version of the "Morse inequality" ($\star$) for cohomology mod p. (Dually, one could work with mod p homology, as in [Eps61].) In fact, as Proposition 4.20 below indicates, the mod p bound is in general weaker (although the two bounds coincide for groups in $\mathcal{H}_p$), so that

$$\operatorname{def}(G) \leq \operatorname{rk}(H_1(G)) - d(H_2(G)) \leq b^1(G, \mathbb{F}_p) - b^2(G, \mathbb{F}_p).$$

Although standard, we include a proof of Proposition 4.20 for completeness and later reference.

Note first that if we write a finite abelian group A in its primary decomposition

$$A = \bigoplus_{i=1}^{k} (\mathbb{Z}/p_i^{\alpha_{i,1}} \oplus \cdots \oplus \mathbb{Z}/p_i^{\alpha_{i,r_i}}) = \bigoplus_{i=1}^{k} A_{p_i}$$

where $p_1, \ldots, p_k$ are distinct primes (and A_{p_i} denotes the Sylow p-subgroup of A), then the rank of A is the maximal rank of its subgroups A_{p_i}, that is, $d(A) = \max_i r_i$. Equivalently,

$$d(A) = \max_i \dim_{\mathbb{F}_{p_i}} (A \otimes \mathbb{F}_{p_i}).$$

If this maximum is attained for a given prime p_i, we say p_i *rank-dominates* A. Every finite abelian group is rank-dominated by one or more primes.

Proposition 4.20. *Let G be a group such that $H_1(G)$ and $H_2(G)$ are finitely generated and let p be a prime. Then*

$$b^1(G, \mathbb{F}_p) - b^2(G, \mathbb{F}_p) \geq \operatorname{rk} H_1(G) - d(H_2(G)) \tag{†}$$

with equality if and only if the torsion subgroup of $H_2(G)$ is rank-dominated by p, that is, $d((H_2G)_{\mathrm{tor}}) = d((H_2G)_{\mathrm{tor}} \otimes \mathbb{F}_p)$.

Proof. Let

$$H_1(G) \cong \mathbb{Z}^r \oplus T_1$$
$$H_2(G) \cong \mathbb{Z}^s \oplus T_2$$

where T_1 and T_2 are finite. By the Universal Coefficient Theorem

$$\begin{aligned} H^1(G, \mathbb{F}_p) &\cong \mathrm{Hom}(H_1(G), \mathbb{F}_p) \\ &\cong \mathbb{F}_p^r \oplus (T_1 \otimes \mathbb{F}_p) \\ H^2(G, \mathbb{F}_p) &\cong \mathrm{Hom}(H_2(G), \mathbb{F}_p) \oplus \mathrm{Ext}(H_1(G), \mathbb{F}_p) \\ &\cong \mathbb{F}_p^s \oplus (T_2 \otimes \mathbb{F}_p) \oplus (T_1 \otimes \mathbb{F}_p) \end{aligned}$$

since Ext is biadditive, $\mathrm{Ext}(\mathbb{Z}, \mathbb{F}_p) \cong 0$ and $\mathrm{Ext}(\mathbb{Z}/n, \mathbb{F}_p) \cong \mathbb{Z}_n \otimes \mathbb{F}_p$ for all n.

The rank $d(\mathbb{Z}^s \oplus T_2) = s + d(T_2)$, and $d(T_2) \geq \dim_{\mathbb{F}_p}(T_2 \otimes \mathbb{F}_p) = b^1(T_2, \mathbb{F}_p)$ with equality if and only if T_2 is rank-dominated by p. Thus

$$\begin{aligned} b^1(G, \mathbb{F}_p) - b^2(G, \mathbb{F}_p) &= (r + b^1(T_1, \mathbb{F}_p)) - (s + b^1(T_2, \mathbb{F}_p) + b^1(T_1, \mathbb{F}_p)) \\ &\geq r - (s + d(T_2)) \\ &= \mathrm{rk}\, H_1(G) - d(H_2(G)). \qquad \square \end{aligned}$$

Remark 4.21. We can prove the specific "Morse inequality" $\mathrm{def}(G) \leq b^1(G, \mathbb{F}_p) - b^2(G, \mathbb{F}_p)$ alone, similarly to the proof of ($\star$). Suppose that P is a presentation of G, and let K_P be the corresponding presentation complex. Then $\mathrm{def}\, P = 1 - \chi(K_P) = b^1(K_P, \mathbb{F}_p) - b^2(K_P, \mathbb{F}_p)$. Attaching cells of dimension 3 and higher to K_P to construct a classifying space for G will only have the effect of possibly killing off some of $H^2(K_P, \mathbb{F}_p)$. Thus $b^1(K_P, \mathbb{F}_p) - b^2(K_P, \mathbb{F}_p) \leq b^1(G, \mathbb{F}_p) - b^2(G, \mathbb{F}_p)$, which completes the proof.

It now makes sense to introduce the following definition.

Definition 4.22. A finitely presented group G is called *p-efficient* if it has deficiency $b^1(G, \mathbb{F}_p) - b^2(G, \mathbb{F}_p)$.

Remark 4.23. Every p-efficient group is efficient. Conversely, every efficient group is p-efficient *for some* p. This is because the torsion subgroup of $H_2(G)$ will be rank-dominated by some p, giving equality in (†).

Remark 4.24. Since $b^1(G, \mathbb{F}_p)$ gives a lower bound on the rank $d(G)$, an alternative definition of $\mathcal{H}_p$ would be "the class of p-efficient groups admitting a minimal presentation".

The fact that $\mathcal{H}$ is closed under direct products, with deficiency determined by the number of generators and relators in minimal efficient presentations, can be proved almost identically to Lemma 4.4, except that one uses the corresponding Künneth formula for mod p cohomology instead.

We can also remove the mention of $\mathbb{F}_p$ coefficients entirely (hiding it away in Universal Coefficient Theorem calculations as in the proof of Proposition 4.20) and identify $\mathcal{H}_p$ as follows:

Proposition 4.25. *$\mathcal{H}_p$ coincides with the class of groups G such that*

- *G admits a minimal, efficient presentation;*
- *G has rank $d(G) = d(H_1(G))$;* and
- *the torsion subgroups of $H_1(G)$ and $H_2(G)$ are both rank-dominated by p.*

Such a group is p-efficient (since p rank-dominates torsion in $H_2(G)$ and G is efficient) and the last two points together imply that $d(G) = b^1(G, \mathbb{F}_p)$, so that a minimal presentation for G has the correct number of generators and relators.

It is now clear that $\mathcal{H}_p$ includes all of $\mathcal{G}_p$ (of course, this also follows straight from Definition 4.19 via Universal Coefficient Theorem calculations): every group in $\mathcal{G}_p$ has a minimal, efficient presentation, its rank is preserved in abelianization, and the homology groups are p-groups so certainly are rank-dominated by p.

Example 4.26. $\mathbb{Z}/2 \times \mathbb{Z}/3 \in \mathcal{H}_2$ but $\mathbb{Z}/2 \times (\mathbb{Z}/3)^2 \notin \mathcal{H}_2$.

As a particular example of our above criteria (Proposition 4.25), the surface group Σ_g, which has $H_1(\Sigma_g) \cong \mathbb{Z}^{2g}$, $H_2(\Sigma_g) \cong \mathbb{Z}$, and admits the presentation

$$\Sigma_g \cong \langle\, a_1, b_1, \ldots, a_g, b_g \mid [a_1, b_1] \cdots [a_g, b_g] \,\rangle,$$

is an element of $\mathcal{H}_p$, for every prime p.

4.6.1 Deficiencies of infinite Kähler groups

The closure property of $\mathcal{H}_p$ was used by Kotschick [Kot12] to construct Kähler groups of all but finitely many negative deficiencies. With our basic p-groups in hand, as the class of Kähler groups is closed under direct product, it turns out to be much simpler to construct Kähler groups of arbitrary negative deficiency than specifically finite (hence Kähler) groups of arbitrary negative deficiency. The shortcut is as follows:

Example 4.27. Recall that the finite p-groups A_p and C_p admit minimal efficient presentations with the number of generators and relators being $(2, 2)$ for A_p and $(1, 1)$

for C_p. The minimal efficient presentation for Σ_g has $2g$ generators and 1 relator. Thus

$$\begin{aligned} \operatorname{def}(\Sigma_g \times A_p) &= (2g+2) - (1+2+(2g)2) = -1-2g \quad \text{and} \\ \operatorname{def}(\Sigma_g \times C_p^2) &= (2g+2) - (1+3+(2g)2) = -2-2g. \end{aligned}$$

In other words, every negative integer is the deficiency of a group which has a finite index surface subgroup.

4.7 Minimal presentations of non-efficient groups

The purpose of this section is to demonstrate that minimality and efficiency of presentations are independent concepts. Recall that a presentation of G is minimal if the number of generators is $d(G)$ and the number of relators is $d(G) + \operatorname{def}(G)$. It is open whether every group admits a minimal presentation. Lustig gave the first example of a torsion-free non-efficient group [Lus95], namely $\mathbb{Z} \times T_{2,3}$, where $T_{2,3} = \langle\, x, y \mid x^2 = y^3 \,\rangle$ is the fundamental group of the trefoil knot complement. The obvious presentation

$$L = \langle\, x, y, z \mid x^2 = y^3,\ [x,z],\ [y,z] \,\rangle$$

of the direct product achieves the group's deficiency, namely 0, however the homological bound $\operatorname{rk}(H_1(G)) - d(H_2(G)) = 1$. (This bound is easily computed using the direct product of the presentation complex of the one-relator group $T_{2,3}$ as its classifying space, via Lyndon's theorem, and S^1. Lustig's achievement was to show that no presentation of the group realizes this bound.)

As we are not aware of it having being noted in the literature, we include here (with proof) a minimal presentation of this group.

Proposition 4.28. *Lustig's non-efficient group admits the minimal presentation*

$$M = \langle\, a, b \mid [a, b^3],\ [a^2, b] \,\rangle.$$

Proof. Let

$$\begin{aligned} \varphi\colon M &\to L & \qquad \psi\colon L &\to M \\ a &\mapsto xz & x &\mapsto a^3b^{-3} \\ b &\mapsto yz & y &\mapsto a^2b^{-2} \\ & & z &\mapsto a^{-2}b^3. \end{aligned}$$

These are well-defined group homomorphisms: $\varphi([a,b^3]) = [xz,(yz)^3] = [x,y^3] = 1$ since z is central, and $y^3 = x^2$ commutes with x, and similarly $\varphi([a^2,b]) = [(xz)^2,xy] = [x^2,y] = [y^3,y] = 1$. In the other direction, as a^2 and b^3 are central in M,

$$\psi(x^2y^{-3}) = (a^3b^{-3})^2(a^2b^{-2})^{-3} = a^6b^{-6}b^6a^{-6} = 1$$

and $\psi(z)$ is central so $\psi([x,z]) = \psi([y,z]) = 1$.

Now $\psi(\varphi(a)) = a^3b^{-3}a^{-2}b^3 = a$, $\psi(\varphi(b)) = a^2b^{-2}a^{-2}b^3 = b$, again since a^2 and b^3 are central, so $\psi \circ \varphi = \mathrm{id}_M$ and thus φ is injective.

We have $\varphi(a^{-2}b^3) = (xz)^{-2}(yz)^3 = x^{-2}y^3z = z$, so since xz and yz are also in the image of φ, it is clear that it is surjective. Thus M and L are isomorphic. □

Remark 4.29. Groups which are shown to be non-efficient using Lustig's criterion, like the above example, cannot have a relation gap (see Section 1.4.3), as noted in [BT07, Proposition 5].

4.8 A Schur multiplier computation

The object of study in this section is the group

$$\mathrm{J}_p := \langle\, a,b \mid a^{p^2},\, b^{p^2},\, [[a,b],a],\, [[a,b],b],\, [a,b]^p \,\rangle$$

as used in one of the variations on our construction, Lemma 4.16. It is the smallest p-group with a minimal efficient presentation that is 2-generator 5-relator. We compute the Schur multiplier of J_p in two independent ways.

Proposition 4.30. *The Schur multiplier of* J_p *is* $(\mathbb{Z}/p)^3$.

In both computations, we restrict to the case of odd p for simplicity. For the specific case $p = 2$, one could refer to standard tables, for example in [CTVZ03], or ask GAP to compute `AbelianInvariantsMultiplier(SmallGroup(32, 2))`. We will suppress the subscript and simply write $\mathrm{J} = \mathrm{J}_p$.

4.8.1 Via the Hopf formula

We exploit the fact that the Schur multipliers of unitriangular groups have been computed.

The group $UT_3(\mathbb{Z}/p^2)$ is isomorphic to $\widetilde{\mathrm{J}} := \langle\, a, b \mid a^{p^2},\ b^{p^2},\ [[a,b],a],\ [[a,b],b] \,\rangle$. We see this by observing that

$$a \mapsto \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, b \mapsto \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$$

defines a surjective homomorphism $\widetilde{\mathrm{J}} \to UT_3(\mathbb{Z}/p^2)$. This homomorphism is injective as $\widetilde{\mathrm{J}}$ must have order at most p^6: since it is nilpotent of class 2, every element is expressible as $a^\alpha b^\beta [a,b]^\gamma$ and $[a,b]^{p^2} = [a^{p^2}, b] = [1,b] = 1$, so we can assume $0 \le \alpha, \beta, \gamma < p^2$.

The centre of $\widetilde{\mathrm{J}}$ is of order p^2 and generated by $[a,b]$. So J is simply the quotient of $\widetilde{\mathrm{J}} \cong UT_3(\mathbb{Z}/p^2)$ by its unique central subgroup of order p.

First Proof of Proposition 4.30. Theorem 1.1 of [Jez14] says, in particular, that the Schur multiplier of $UT_3(\mathbb{Z}/m)$ is $(\mathbb{Z}/m)^2$ for m odd. The case $m = p^2$ gives $H_2(\widetilde{\mathrm{J}}) \cong (\mathbb{Z}/p^2)^2$.

This then gives a useful lower bound on the Schur multiplier of J; a quick examination of the E_2 page of the Lyndon–Hochschild–Serre spectral sequence allows one to prove the following.

Lemma 4.31 (Theorem 4.1 (i) in [Jon73])**.** *Let G be a finite group and Z a central subgroup. Set $Q = G/Z$. Then $|H_2(G)||G' \cap Z|$ divides $|H_2(Q)||H_2(Z)||H_1(Q) \otimes Z|$.*

Applying the lemma to $G = \widetilde{\mathrm{J}}$ and central $Z \cong \mathbb{Z}/p$, so that $Q = \mathrm{J}$, gives $|H_2(\mathrm{J})| \ge p^3$.

We now prove that $H_2(\mathrm{J})$ is exponent p. Let $\mathrm{J} = F/R$ be the above presentation, so $F = F(a,b)$ and $R = \langle\!\langle a^{p^2}, b^{p^2}, [[a,b],a], [[a,b],b], [a,b]^p \rangle\!\rangle$. Hopf's formula for the Schur multiplier says that

$$H_2(\mathrm{J}) \cong ([F,F] \cap R)/[F,R] = \ker(R/[F,R] \xrightarrow{\varphi} F/[F,F]).$$

The five normal generators for R will be generators for the abelian quotient $R/[F,R]$. The first two, namely a^{p^2} and b^{p^2}, generate a subgroup $I \le R/[F,R]$ mapped injectively by φ to $F/[F,F]$, with image $p^2\mathbb{Z} \times p^2\mathbb{Z} \le \mathbb{Z} \times \mathbb{Z}$. The last three, namely $[[a,b],a]$, $[[a,b],b]$ and $[a,b]^p$, are all in the kernel of φ. Thus $R/[F,R] = I \oplus \ker\varphi$ and the kernel is generated by these last three generators. It remains to show that they all have order p in $F/[F,R]$.

We recall the well-known and easily checked fact that, in general, if $[x,y]$ is central, then $[x,y]^n = [x^n, y]$. Since $[[a,b],a] \in R$, it is central in $F/[F,R]$. Thus

$[[a,b],a]^p = [[a,b]^p,a] = 1$ in $F/[F,R]$, as $[a,b]^p \in R$. Similarly $[[a,b],b]^p = 1$ in $F/[F,R]$.

Recall furthermore that if $[y,x]$ is central, then $(xy)^n = x^n y^n [y,x]^{\binom{n}{2}}$. Setting $x = a$, $y = [a,b]$ and $n = p^2$, we have $(a[a,b])^{p^2} = a^{p^2}[a,b]^{p^2}[[a,b],a]^{\binom{p^2}{2}}$ and since $a[a,b] = b^{-1}ab$ this yields

$$[a^{p^2}, b] = [a,b]^{p^2}[[a,b],a]^{\binom{p^2}{2}}.$$

Now $a^{p^2} \in R$ so $[a^{p^2}, b] = 1$ in $F/[F,R]$. Since $\binom{p^2}{2}$ is a multiple of p, we see that $[[a,b],a]^p = 1$ implies $[[a,b],a]^{\binom{p^2}{2}} = 1$ in $F/[F,R]$ as well. Hence $([a,b]^p)^p = 1$ as required.

As $H_2(\mathrm{J}) \cong \ker\varphi$ has order at least p^3 and is generated by three elements of order p, it must in fact be isomorphic to $(\mathbb{Z}/p)^3$. $\square$

4.8.2 Via the LHS spectral sequence

For convenience, we compute cohomology with $\mathbb{F}_p$ coefficients (from which the rank of the Schur multiplier can be deduced) using known descriptions of the differentials d_2 and d_3 for this setting as described in [Lea93]. (That paper in fact computes cohomology rings for all central extensions of C_3 by a 2-generated abelian 3-group; here we are going through the details explicitly.) This approach exploits the ring structure on cohomology and uses the fact that the extension class is a cup product of two 1-cocycles.

Second Proof of Proposition 4.30. Throughout we will use the short exact sequence for J corresponding to abelianization, namely

$$1 \to \mathbb{Z}/p \to \mathrm{J} \to \mathbb{Z}/p^2 \times \mathbb{Z}/p^2 \to 1.$$

We write $Q = \mathbb{Z}/p^2 \times \mathbb{Z}/p^2$. This exhibits J as a *central* extension of $\mathbb{Z}/p$ by Q.

By the Universal Coefficient Theorem,

$$H^2(G, \mathbb{F}_p) \cong H_1(G) \otimes \mathbb{Z}/p \oplus H_2(G) \otimes \mathbb{Z}/p$$

for any finite group G. Thus $H^2(G, \mathbb{F}_p)$ will not in general allow us to compute $H_2(G)$, but it will let us establish p-efficiency and thus efficiency for our presentation. However, for the sake of completeness, we again determine the full Schur multiplier of J.

Once we know that $H^2(\mathrm{J}, \mathbb{F}_p) \cong \mathbb{F}_p^5$, it follows that $H_2(\mathrm{J}, \mathbb{Z}) \cong (\mathbb{Z}/p)^3$ by an easy computation on the E^2 page of the Lyndon–Hochschild–Serre spectral sequence for $H^*(\mathrm{J}, \mathbb{Z})$. This is just as in Lemma 4.31, but we include the relevant page, Figure 4.5, for illustration and for the enjoyment of the reader. For the spectral sequence to correctly yield $H_2(\mathrm{J})$, we see that $d^2_{2,0}$ must be a surjection, and thus $E^3_{2,0}$ has order at most p, so the Schur multiplier $H_2(\mathrm{J}, \mathbb{Z})$ has order at most p^3. Since $H^2(\mathrm{J}, \mathbb{F}_p) \cong \mathbb{F}_p^5$ gives $H_2(\mathrm{J}, \mathbb{F}_p) \otimes \mathbb{F}_p \cong \mathbb{F}_p^3$, the Schur multiplier must in fact be an elementary abelian p-group (a vector space over $\mathbb{F}_p$) as otherwise its order would be too large.

$$
\begin{tikzcd}
 & 2 & 0 & 0 & 0 \\
H_*(\mathbb{Z}/p, \mathbb{Z}) & 1 & \mathbb{Z}/p & (\mathbb{Z}/p)^2 & \mathbb{Z}/p \\
 & 0 & \mathbb{Z} & \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2 & \mathbb{Z}/p^2 \arrow[ull, two heads] \\
 & & 0 & 1 & 2 \\
E^2 & & & H_*(Q, \mathbb{Z}) &
\end{tikzcd}
$$

Figure 4.5: The E^2 page of the LHS spectral sequence for $H_*(\mathrm{J}, \mathbb{Z})$

Now we need only compute $H^2(\mathrm{J}, \mathbb{F}_p)$ via the Lyndon–Hochschild–Serre spectral sequence. We recall some basic features of our set up, as in [Lea93]. The cohomology ring is described by

$$H^*(\mathbb{Z}/p^k) \cong \mathbb{F}_p[u, t]/I,$$

where u is degree 1, t is degree 2, and the ideal I is generated by u^2. In particular, $H^d(\mathbb{Z}/p^k, \mathbb{F}_p) \cong \mathbb{F}_p$ in every degree d. Now we focus on the kernel of the extension, so $k = 1$. Since our extension is central, Q acts trivially on $H^*(\mathbb{Z}/p, \mathbb{F}_p)$ and thus the E_2 page of the spectral sequence is simply the tensor product of $H^*(\mathbb{Z}/p, \mathbb{F}_p)$ with $H^*(Q, \mathbb{F}_p)$. The relevant terms are indicated in Figure 4.6. They are easily computed via the Künneth formula, together with the cohomology ring of a cyclic p-group described above.

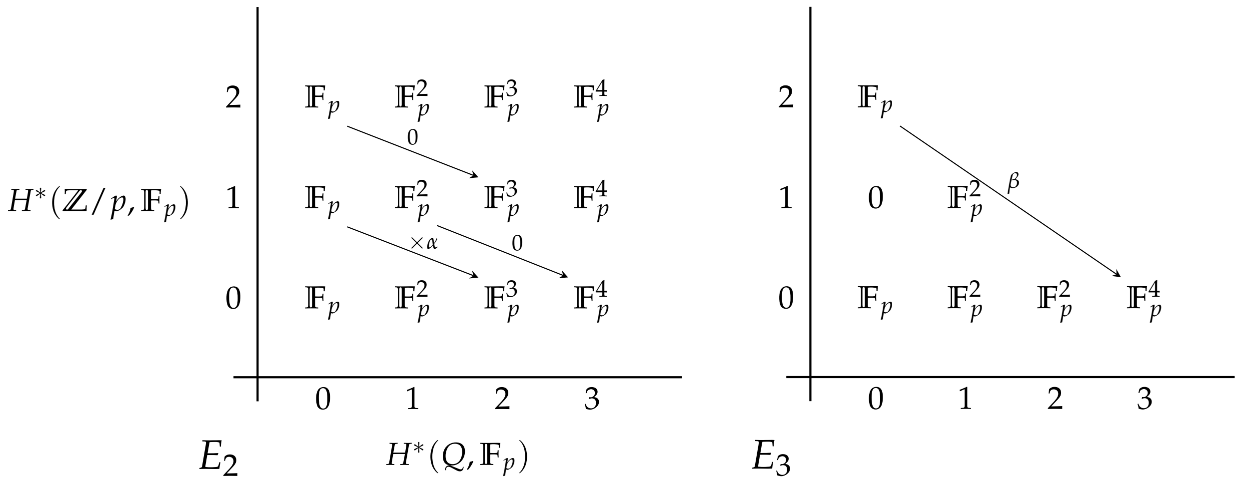


Figure 4.6: The E_2 and E_3 pages of the LHS spectral sequence for $H^*(\mathrm{J}, \mathbb{F}_p)$

There are, modulo choice of isomorphisms, explicit descriptions of the differentials that we need: $d_2(u)$ is the extension class $\alpha \in H^2(Q, \mathbb{Z}/p)$ corresponding to our group J, $d_2(t) = 0$, and $d_3(t)$ is the Bockstein $\beta(\alpha)$ for the short exact sequence of coefficients $1 \to \mathbb{F}_p \to \mathbb{Z}/p^2 \to \mathbb{F}_p \to 1$.

As our extension class is non-zero (that is, the extension is not split), $d_2 \colon E_2^{0,1} \to E_2^{2,0}$ is an inclusion, indicated by $\times\alpha$ in the figure. We can determine $d_2 \colon E_2^{1,1} \to E_2^{3,0}$ by exploiting the ring structure of $H^*(Q, \mathbb{F}_p)$ and the fact that the E_2 is a tensor product. Let ζ and η be the generators of $H^1(Q, \mathbb{F}_p)$ coming from the two copies of $H^1(\mathbb{Z}/p^2, \mathbb{F}_p)$ corresponding to decomposition $Q = \mathbb{Z}/p^2 \oplus \mathbb{Z}/p^2$. Then $E_2^{1,1}$ has the basis $u \otimes \zeta$ and $u \otimes \eta$. As $d_2(\zeta) = d_2(\eta) = 0$ (the differential $d_2^{1,0}$ goes off the first quadrant) we see that the image of $d_2^{1,1}$ will be generated by $\alpha \smile \zeta$ and $\alpha \smile \eta$.

Now we claim that the extension class $\alpha = \zeta \smile \eta$. This is easy to check, as the factor set given by the obvious choice of set-theoretic section of the extension — namely, $(s,t) \mapsto a^s b^t$ — gives the 2-cocycle $Q \times Q \to \mathbb{Z}/p : ((s,t),(s',t')) \mapsto s't$. This is exactly the cup product of the generators ζ and η of the two tensor factors of $H^1(Q, \mathbb{F}_p)$.

So now, just as $u^2 = 1$, we have $\zeta^2 = \eta^2 = 0$, so $d_2^{1,1}$ is the zero map.

We can now turn the page to E_3, with the terms indicated in Figure 4.6. It remains to determine the Bockstein. We need to prove that the Bockstein is zero, so that $E_3^{2,0} = E_\infty^{2,0} = \mathbb{F}_p$ and thus $H^2(\mathrm{J}, \mathbb{F}_p) = \mathbb{F}_p^5$.

The short exact sequence

$$1 \to \mathbb{Z}/p \xrightarrow{\iota} \mathbb{Z}/p^2 \xrightarrow{\pi} \mathbb{Z}/p \to 1$$

induces the long exact sequence

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & H^2(Q, \mathbb{Z}/p) & \xrightarrow{\iota_*} & H^2(Q, \mathbb{Z}/p^2) & \xrightarrow{\pi_*} & H^2(Q, \mathbb{Z}/p) & \xrightarrow{\beta} & H^3(Q, \mathbb{F}_p) & \longrightarrow \cdots \\
& \cong & & \cong & & \cong & & \cong & \\
& (\mathbb{Z}/p)^3 & & (\mathbb{Z}/p^2)^3 & & (\mathbb{Z}/p)^3 & & (\mathbb{Z}/p)^4 &
\end{array}
$$

which can only be exact at $H^2(Q, \mathbb{Z}/p^2)$ if ι_* is injective and π_* is surjective, whence the Bockstein $\beta = 0$. $\square$

Chapter 5

Profinite rigidity in the SnapPea census

It's so funky and it's low volume.

Jack Stratton

5.1 Introduction

A standard approach to studying infinite groups is through their finite quotients. While this has serious limitations in general – exemplified by the existence of infinite groups having *no* non-trivial finite quotients – in many contexts, the finite quotients of a group encode much important information about it. For instance, the fundamental group of any compact 3-manifold is *residually finite*, so it has enough finite quotients that every non-trivial element survives in one. The question of how much is encoded in the finite quotients of a 3-manifold group has gathered much attention in recent years. One well-known open question, attributed to Long and Reid in [Ago14, Question 1], is the following:

Question 5.1. Let M and N be finite volume hyperbolic 3-manifolds. If $\pi_1 M$ and $\pi_1 N$ have the same finite quotients, does this imply that $\pi_1 M \cong \pi_1 N$?

By Mostow Rigidity, $\pi_1 M \cong \pi_1 N$ implies that M and N are isometric.

It is convenient to collect the totality of the finite quotients of a (finitely generated) group G into a single algebraic object, namely its *profinite completion* $\widehat{G}$, the inverse limit of the inverse system of its finite quotients, as we introduced in Section 1.4.2. This topological group determines the set of (isomorphism classes of) finite quotients, and a well-known result, Proposition 1.12, says that, conversely, it is determined by the set of finite quotients if it is (topologically) finitely generated.

There has been a lot of recent progress in the study of profinite properties of 3-manifolds, with results showing both rigidity and flexibility. Various properties of 3-manifolds have been shown to be profinite invariants (that is, determined by the profinite completion of the fundamental group), including hyperbolicity, by Wilton–Zalesskii [WZ17], and being fibred, by Jaikin-Zapirain [Jai17] following [BR15], [BRW17] and [BF15]. The most significant progress on Question 5.1 is a theorem of Bridson, Reid and Wilton [BRW17], proving that it holds in the case that M is a once-punctured torus bundle over the circle (so that $\pi_1 M \cong F_2 \rtimes \mathbb{Z}$), building on earlier work that did special cases [BR15; BF15]. The forthcoming paper [BMRS17] gives the first examples of groups which are large (in the sense of virtually surjecting onto a non-abelian free group) and are *absolutely profinitely rigid*, by which we mean that they are each uniquely determined by their profinite completion amongst all finitely generated residually finite groups. The examples are namely $\mathrm{PGL}(2, \mathbb{Z}[\omega])$ and $\mathrm{PSL}(2, \mathbb{Z}[\omega])$, where ω is a cube root of unity, which are fundamental groups of hyperbolic 3-orbifolds, both with the figure eight knot complement as a finite sheeted cover.

The purpose of this chapter is to report on a computational proof that the manifolds in the benchmark census of (low) finite volume hyperbolic 3-manifolds have pairwise non-isomorphic profinite completions.

Theorem I. *The 72 942 finite volume hyperbolic 3-manifolds in the SnapPea census are distinguished from each other by the finite quotients of their fundamental groups.*

These census manifolds are those included in the package `SnapPy` [CDGW], of which 11 031 are closed (available in `OrientableClosedCensus`) and 61 911 are cusped (available in `OrientableCuspedCensus`). The cusped examples represent all orientable cusped hyperbolic manifolds that can be triangulated with at most 9 ideal tetrahedra.

Note that Theorem I does not however answer Question 5.1 in the case that we fix M to lie in the census; our computational method can only prove *relative* profinite rigidity within the census, where *both* M and N must be chosen from the census manifolds.

The value of Theorem I goes beyond the achievement of providing the first wholesale evidence for a positive answer to Question 5.1. We mention here one related conjecture, and one direct consequence.

$$
\begin{tikzcd}
 & \pi_1 M \cong \pi_1 N \arrow[rr, Rightarrow] & & \widehat{\pi_1 M} \cong \widehat{\pi_1 N} \arrow[d, Rightarrow] \\
 & & & H_1\,\mathrm{FC}(M) \cong H_1\,\mathrm{FC}(N) \arrow[dd, Rightarrow] \\
M \cong N \arrow[r, Rightarrow] \arrow[uur, Rightarrow] & \mathrm{vol}(M) = \mathrm{vol}(N) & & \\
 & & & \mathrm{FIA}(M) = \mathrm{FIA}(N) \arrow[dl, Rightarrow] \\
 & & \displaystyle \limsup_{[M:\widehat{M}]\to\infty} \frac{\log|(H_1\widehat{M})_{tor}|}{\mathrm{vol}(\widehat{M})} = \limsup_{[N:\widehat{N}]\to\infty} \frac{\log|(H_1\widehat{N})_{tor}|}{\mathrm{vol}(\widehat{N})} \arrow[uul, Rightarrow, "???"'] &
\end{tikzcd}
$$

Figure 5.1: Implications for M and N finite volume hyperbolic 3-manifolds

Asymptotic Volume Conjecture

First, Question 5.1 fits into an important circle of deep work, culminating in the Asymptotic Volume Conjecture following Lück, Bergeron, Venkatesh, Lê, and others.

Conjecture 5.2. *Let M be a finite volume hyperbolic 3-manifold with fundamental group G. Then*

$$\limsup_{[G:K]\to\infty} \frac{\log|(H_1(K,\mathbb{Z}))_{tor}|}{[G:K]} = \frac{\mathrm{Vol}(M)}{6\pi}.$$

The upper bound on the limit has been proved by Lê [Lê14].

One can gradually weaken invariants of manifolds, leading to the spiral of implications shown in Figure 5.1. For a manifold M, let $\mathrm{FC}(M)$ denote the lattice of finite sheeted covers of M. We can apply the homology functor (with trivial $\mathbb{Z}$ coefficient) to get $H_1\,\mathrm{FC}(M)$, a lattice of abelian groups, which we consider to be annotated by the degrees of the corresponding covers. A standard fact, which we recounted in Section 1.4.2, says that $\widehat{\pi_1 M}$ determines $H_1\,\mathrm{FC}(M)$. If we then forget the lattice information, and the number of subgroups of a given index with a given abelianization, we have

$$\mathrm{FIA}(M) := \left\{ ([H_1\check{M}], n) \,:\, n = [M:\check{M}] < \infty \right\}.$$

That is, $\mathrm{FIA}(M)$ is the set of (isomorphism classes of) abelianizations of finite index subgroups of $\pi_1 M$, together with their indices in $\pi_1 M$. Many results are consistent with the aphorism that homology of finite sheeted covers of hyperbolic 3-manifolds tell us almost anything we could wish to know about them. The logical extreme is the following strengthening of Question 5.1.

Question 5.1′. *Let M and N be two finite volume hyperbolic 3-manifolds, and suppose that* $\mathrm{FIA}(M) = \mathrm{FIA}(N)$. *Must M and N be isometric?*

It appears that verifying this conjecture in the SnapPea census with current software is infeasible; see Remark 5.6 below.

The implications in the figure are drawn as a spiral because, conjecturally, we end up back where we started up to "finite ambiguity", since only finitely many hyperbolic 3-manifolds can have the same finite volume [Thu82, 3.6 Theorem].

We note that $\mathrm{FIA}(M)$ is a strictly weaker invariant than $\widehat{\pi_1(M)}$, even when M is a compact 3-manifold (although conjecturally this cannot happen in the hyperbolic case). The following example is due to Gareth Wilkes.

Example 5.3. There are Seifert fibred 3-manifolds M and N such that $\mathrm{FIA}(M) = \mathrm{FIA}(N)$ but $\widehat{\pi_1 M} \not\cong \widehat{\pi_1 N}$. Indeed, the fundamental groups

$$\pi_1 M = \langle\, a, b, c, h \mid h \text{ central},\ a^4 h,\ b^4 h,\ c^2 h,\ abc \,\rangle$$
$$\pi_1 N = \langle\, a, b, c, h \mid h \text{ central},\ a^4 h^3,\ b^4 h^3,\ c^2 h,\ abch \,\rangle$$

are distinguished by their maximal 2-class 6 quotients (of order 2^{12}). These manifolds are commensurable.

As these are Seifert fibred 3-manifolds, we can apply [Wil17, Theorem 1.2] to conclude *a priori* that $\pi_1 M \not\cong \pi_1 N$ implies that $\widehat{\pi_1 M} \not\cong \widehat{\pi_1 N}$.

No duplicates in the census

Second, Theorem I gives independent verification that the census does not contain any duplicates. The standard way of verifying this is to compute the canonical Epstein–Penner cell decomposition. However, rounding errors in imprecise computational arithmetic of real numbers has previously lead to duplicates. One such pair was identified by Burton [Bur14]. Our verification – while of course dependent on many large computer calculations that cannot be replicated by hand – involves only precise discrete computations, in combinatorial group theory and linear algebra over $\mathbb{Z}$.

5.2 Theory

The consequence of Proposition 1.15 is that, given a finite set of finitely presented groups which we believe to be profinitely rigid, there is a naive algorithm that will prove this if it is the case. However, in general we cannot give any prediction as to how long such a verification would take: any computable upper bound on the time needed to determine a *yes* answer allows us to produce a *no* answer, once the allowed time has been exceeded. Indeed, there is a common expectation that the time needed to prove Theorem I would be astronomical (for instance, the 150 groups with trivial abelianization were not distinguished from each other by counting maps onto finite simple groups after several weeks of computation). However, structure theory of the profinite completion and its subgroups and some theory of hyperbolic 3-manifolds reveal why a less naive approach (not simply enumerating maps to finite groups) should be feasible, as our computations have demonstrated.

5.2.1 Hyperbolic 3-manifolds

Simple quotients have previously proved to be effective at distinguishing profinite completions of groups. This was a natural place to look in work on parafree groups of Baumslag–Cleary–Havas [BCH04], because nilpotent quotients cannot distinguish those groups. For the hyperbolic 3-manifolds in the SnapPea census, nilpotent quotients are often similarly ruled out because $b_1(M, \mathbb{Z}/p) \leq 1$ for all primes in many cases (for example, in the case of one cusp, which includes all knot complements, where $H_1(M, \mathbb{Z}) \cong \mathbb{Z}$), which means no non-abelian nilpotent quotients. The profinite completion already encodes all the abelian quotients.

A further theoretical justification for using simple groups is that Long–Reid proved that finite volume hyperbolic 3-manifolds are residually simple [LR98].

Dunfield and Thurston tabulated for each finite simple group of order up to 32 736 the percentage of closed census manifolds with that simple group as a quotient of the fundamental group [DT03, Table 2, p. 12]. Amongst these finite simple groups and these manifolds, a random manifold has a random simple group as a quotient with probability 34.7%. The Mathieu group M_{11} is a quotient of only 17.1% of the manifolds, which is minimal for the simple groups considered.

Note that the achievement of Bridson–Reid–Wilton was to show that the set of groups $F_2 \rtimes \mathbb{Z}$ is profinitely rigid. It is *not* known whether extensions of non-abelian free groups by cyclic groups are profinitely rigid in general. Many examples, in fact more than 47% of the census manifolds, are free-by-cyclic (as determined by Brown's

Criterion, see Remark 1.5). The resolution of the Virtual Fibering Conjecture by Agol, following work of Wise and coauthors, means that any finite volume cusped hyperbolic 3-manifold is virtually free-by-cyclic.

A more ambitious variant of Question 5.1 would be to allow one of the two groups to be any finitely generated residually finite group, that is, to ask whether finite volume hyperbolic 3-manifolds are absolutely profinitely rigid; it is open whether this greatly strengthened conjecture holds. Remesslennikov's question of whether free groups are distinguished in the class of all finitely generated residually finite groups by their profinite completions also remains open.

5.3 Practice

5.3.1 Heuristics

Two strategies to distinguish profinite completions are

- find abelianizations of finite index subgroups; *and*
- count maps onto finite (simple) groups (up to automorphisms of the quotient).

By factoring out automorphisms of the quotient, we are counting the normal subgroups which give the specified finite group as quotient.

One would hope to distinguish hyperbolic 3-manifolds using FIA. Certainly, it appears experimentally that this approach is more effective that enumerating finite quotients. This mimics the phenomenon of Dunfield–Thurston [DT03], where it was easier to verify the virtual Haken conjecture by verifying a stronger, more algebraic, result, namely virtual positive first Betti number, which also involves abelianizations of finite index subgroups. A fundamental difference in character between the virtual Haken conjecture and Question 5.1 is that one only needs to exhibit a Haken cover to prove the conjecture in a specific instance. In our present case, not only are we unable to prove profinite rigidity relative to the class of all finite volume hyperbolic 3-manifold groups, it is difficult to imagine an easily verifiable certificate just within the census, because distinguishing profinite completions requires proving *non-existence* of certain quotients or certain abelianized subgroups, and it is not at all clear how one might do this without repeating the exhaustive enumeration.

However, enumerating all subgroups of index n has complexity factorial in n, and many manifolds have the same abelianizations of low index subgroups.

We thus turn to a powerful combination of the two naive strategies: determining maps onto finite simple groups, then computing the abelianization of the kernel. There is a very good heuristic reason for why this should be effective; we quantify this using entropy, the expected information in a random variable.

Definition 5.4. Let X be a discrete random variable taking values in $\{x_1, \ldots, x_n\}$, each with probability $P(x_i)$. The *entropy* of X is

$$H(X) := -\sum_{i=1}^{n} P(x_i) \log_2 P(x_i).$$

The maximal entropy of a random variable taking n values is $\log_2 n$ bits, and occurs when it is the uniform distribution (each value occurs with probability $\frac{1}{n}$). Our random variables will be the value(s) of invariants for the $N = 72942$ census manifolds, drawn uniformly. The entropy is $\log_2 N$ if and only if a given set of invariants completely distinguishes the manifolds.

The number of maps from a group G defined by a k generator presentation onto a finite group Q of order n is certainly at most n^k. (For non-abelian simple groups, with few outer automorphisms and very high probability that a random pair generates, n^{k-1} is a fairly good approximation.) Thus, computing the number of maps to a finite group can only provide us with approximately $k \log_2 n$ bits of information.

On the other hand, torsion in homology grows very quickly (the torsion-free rank also provides useful, albeit secondary, information). We expect its size to be on the order of $e^{n \operatorname{Vol}(M)/6\pi}$ for a normal subgroup of index n. Thus we get entropy that is linear, rather than logarithmic, in the size of the quotient Q considered, provided the homology groups arising are sufficiently varied. (In considering these heuristic estimates, we must keep in mind that there is a bound of $\log_2 N$ on entropy for any collection of invariants, and that the random variables will not be independent so we cannot simply sum up their respective entropies.) For any given Q, a large but not overwhelming percentage of manifolds will have the same amount of information under both schemes, as they have no surjections onto Q; when a group has more than 1 surjection to Q, this only helps us.

Remark 5.5. There are various ways one could measure how a set of invariants contributes towards distinguishing a finite set of objects. We believe that entropy is the best measure. Simple alternatives, such as counting the number of equivalence classes or the number of objects that have been distinguished from all the others, fail to capture the "shape" of the partition. Consider for example two possible partitions

	entropy of:		when $\#\{K\} \geq 1$:	
group	$\#\{K\}$	$\{H_1 K\}$	entropy	$\log_2$(#manifolds)
A_5	2.37	8.90	12.37	15.51
$\mathrm{PSL}(2,7)$	2.88	10.82	13.98	15.66
A_6	2.80	9.92	13.53	15.57

Table 5.1: Entropy in the number of regular covers K of M with Galois group isomorphic to a given finite simple group, for M in the census; entropy in the (multi)set of abelianizations of such covers; entropy amongst only those manifolds with at least one such simple cover; upper bound on that entropy

of 10 objects, either as 5 pairs or as sextuple and 4 singletons. Entropy captures well the intuitive view that partitioning the set into 5 pairs is better progress.

Table 5.1 lists the entropy of counting maps to some finite simple groups for the sample space of SnapPea census manifolds (taken uniformly at random). The last two columns demonstrate that a lot of the gap between entropy of the homology of covers K and the upper bound of $\log_2(72942)$ is accounted for by the number of manifolds which have no such cover.

We now recount a very concrete example of the power of computing abelianizations of kernels. The two knots identified in SnapPy as `K14a3482` and `K14a3494` are very difficult to distinguish. For full reference, their Dowker–Thistlethwaite codes are

`4 10 14 16 2 24 22 18 8 6 26 28 20 12` and
`4 10 14 16 2 26 24 18 8 6 12 28 20 22`

respectively. Their complements have the same volume 24.1942..., their Alexander polynomials are identical, they have the same Khovanov homology (computed with `KnotKit` [See]), and they have the same number of surjections onto any simple group of order less than 2500. Nonetheless, the abelianization of kernels of maps to the simple group $\mathrm{PSL}(2,7)$ suffices to show that their profinite completions differ.

As well as having high entropy, the kernels (as a set) are characteristic, and the problem of proving non-existence is dissolved into enumerating maps to a finite quotient. Enumerating non-normal subgroups even of index $60 = |A_5|$ is completely infeasible.

Note that the profinite completion does *not* determine the nilpotent quotients of a group [Rem67] (an interesting connection to Chapter 2 is that the set of quotients $G/G^{(m)}$ therefore does not determine G amongst finitely generated nilpotent groups). Thus, a commonly used computational technique for distinguishing groups is not available to us.

5.3.2 Difficulties and limitations

The main concerns at this point are twofold: the gap between predictions of torsion in homology and actual low volume results, and the correlation between different invariants.

A practical difficulty, which we are yet to explain satisfactorily, is that GAP has extreme difficulty computing `GQuotients` on a small number of examples (less than 1 in 1000, at least for the smallest of simple quotients). For instance, the fundamental group G of the manifold identified in SnapPy as `t05599(0,0)` has the presentation

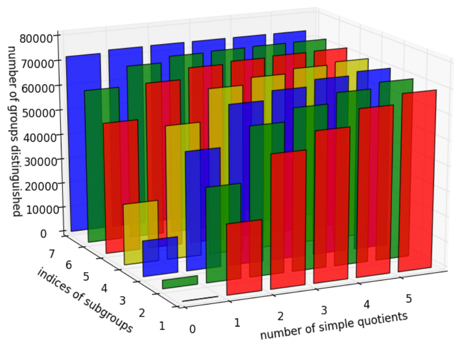
$$\langle\, a, b, c \mid a^2b^5a^2b^2c^{-2},\, a^5c^3b^{-2}\,\rangle.$$

It has one normal subgroup N of index 60 with quotient $G/N \cong A_5$, and $H_1N \cong \mathbb{Z}^{12} \times \mathbb{Z}/12$. Computing all the surjections from G to A_5 with `GQuotients` (which works very directly with the finite presentation) takes GAP 20 minutes and requires gigabytes of memory. We rolled our own method, enumerating all maps from the group G to its quotient Q. The number of iterations we run is bounded by $|\mathrm{Inn}(Q)\backslash Q| \cdot |Q|^{d(G)-1}$, because we can assume without loss of generality that the first generator of G is sent to a preferred element in each conjugacy class, and then send the second generator to a preferred element modulo the centralizer of the image of the first element, and so on. In short, the tuple of images of generators is chosen to be minimal under the lexicographic ordering (after picking an arbitrary order on Q) within its conjugacy class. We simply check at the end which maps give the same kernel; since simple groups have such small outer automorphism groups, we will by this point have only overcounted by a factor of 2 or 4 usually (and it is not worth the hassle of explicitly factoring out by the action of $\mathrm{Aut}(Q)$ and not just $\mathrm{Inn}(Q)$ before this stage). Our method computes all surjections of the aforementioned G onto A_5 in under a second, a speed up of over 1000.

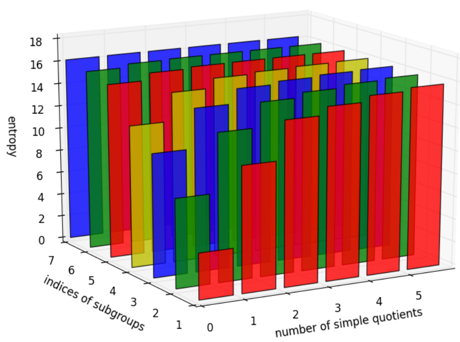
5.3.3 Methods

We computed the minimum number of invariants to distinguish the groups. That is, as soon as a group had been distinguished from the others, it was removed from consideration. We computed

- abelianizations of subgroups up to index 7;



(a) Number of manifolds distinguished



(b) Entropy

Figure 5.2: Information in low-index subgroups and small simple quotients

- the abelianization of the maximal abelian cover if that was finite index, and failing that the cyclic covers up to index 10 if the abelianization was $\mathbb{Z}$; *then*
- abelianizations of kernels of maps to small finite non-abelian simple groups

This was performed in parallel in twenty cores, coordinated by a `python` script. We used the wonderful program `SnapPy` [CDGW] to work with the manifolds in question, and in particular to extract group presentations, and used `GAP` [GAP16] for all the group theoretic computations.

5.4 Results

Theorem I. *The 72 942 finite volume hyperbolic 3-manifolds in the SnapPea census are distinguished from each other by the finite quotients of their fundamental groups.*

This took around 64 hours of CPU time.

A plot indicating the number of manifolds distinguished and the entropy from computing abelianization of

- all subgroups up to a given index, together with
- all kernels of maps onto the smallest 1, 2, 3, 4 or 5 simple groups

is indicated in Figure 5.2. The plots are very similar; we note however that the integral homology of a manifold alone, which only distinguishes 102 or the manifolds, still has approximately 4 bits of information.

Remark 5.6. We were unsuccessful in distinguishing the groups using only FIA. For the 11 031 closed manifolds in the census, two months of CPU time was insufficient. At this point, there were 13 manifolds left to be differentiated, in 5 pairs and 1 triple. No proper subgroups had been found up to index 12. Beyond this index, the exhaustive search for finite index subgroups can take months for a single group. Under the reasonable assumption that these groups do have enough subgroups waiting at index 13, verifying that FIA distinguishes them would take years of CPU time. Thus, it is unlikely that we could succeed at this task in our lifetime without implementing a parallel algorithm for enumerating low-index subgroups.

A very natural question to ask at this point, especially in light of the preceding remark, is: what indices of subgroup are needed, and what order of simple quotients? For instance, a group with no subgroup of index up to 12 cannot map onto any of the 5 simple groups of order less than 1000, because they all have low index subgroups that could be pulled back.

The answer is that the largest simple quotient used was $\mathrm{PSL}(2, 23)$, of order 6072. We tabulate the number of manifolds groups whose profinite completions had been distinguished from all the others at each stage of the computation in Table 5.2. In addition, for the 13 non-abelian simple groups used we list their order and minimal index of a proper subgroup. With the exception of $\mathrm{PSU}(3, 3)$ (which happened to provide no useful information), they all have a subgroup of index 24 or less.

Remark 5.7. This does not mean that all of the 3-manifold groups considered have a subgroup of index 24 or less: we stopped computing subgroups of a group as soon as it was distinguished from all the others. What we can definitely say is that every group has a subgroup of index at most 401, the largest prime p such that there exists a manifold M for which $\mathbb{Z}_p$ is the smallest non-trivial quotient of $H_1 M$. The only such M for $p = 401$ is `v1860(2,3)` (with $H_1 M \cong \mathbb{Z}_{401}$), so we did not need to find any other subgroups of $\pi_1 M$ (it does, however, surject onto $\mathrm{PSL}(2, 14)$, so it has a subgroup of index 14). The last of the 150 perfects groups remaining was the fundamental group of `s636(-4,3)`, which was distinguished by virtue of having no maps onto the non-abelian simple groups up to $\mathrm{PSL}(2, 16)$. We determined separately that it maps onto $\mathrm{PSL}(3, 3)$, so it has a subgroup of index 13 (much less than 401).

Invariant	# dist.
FIA to index 1	102
FIA to index 2	3317
FIA to index 3	10837
FIA to index 4	10095
FIA to index 5	28068
FIA to index 6	9217
FIA to index 7	10029
abelian covers	966

Group	Order	Min. index	# dist.
A_5	60	5	8
$\mathrm{PSL}(2,7)$	168	7	12
A_6	360	6	4
$\mathrm{PSL}(2,8)$	504	9	101
$\mathrm{PSL}(2,11)$	660	11	82
$\mathrm{PSL}(2,13)$	1092	14	51
$\mathrm{PSL}(2,17)$	2448	18	37
A_7	2520	7	0
$\mathrm{PSL}(2,19)$	3420	20	8
$\mathrm{PSL}(2,16)$	4080	17	2
$\mathrm{PSL}(3,3)$	5616	13	2
$\mathrm{PSU}(3,3)$	6048	28	0
$\mathrm{PSL}(2,23)$	6072	24	4

Table 5.2: The number of groups distinguished at each stage, and for the 13 smallest non-abelian simple groups also the order and minimal index of proper subgroups

5.5 Future work

If one were to attempt to distinguish the manifolds using FIA alone, it would be possible to construct a partial certificate, from which one could reliably re-prove the appropriate version of Theorem I more quickly. For instance, if at some point it was necessary to distinguish a group G with a 2-generator presentation and a group H with a 3-generator presentation, a certificate could record a description of a subgroup of H (as stabilizer of a point in a permutation representation), and then one only needs to show that G has no subgroup at that index with that abelianization; this is much faster than enumerating subgroups in H exhaustively, because of the difference in presentation rank.

Many invariants turned out to provide little entropy (or not to help distinguish some subset of the groups at all). An interesting and natural question is: given all the invariants computed, and the times of computation (or re-computation, modulo a potential certificate), what is the subset of minimal computation time that distinguishes the groups? This is in fact an NP-complete problem, as was proved by Hyafil and Rivest [HR76]. However, given that, especially for low index subgroup enumeration, the time required grows quickly, it seems that heuristics would allow for a very good approximation to the optimal binary decision tree.

We will also verify the computations independently with magma.

One could also exploit the lattice structure of FIA, although thus far we have not produced a concrete example where this provides additional information over the multiset of abelianizations together with subgroup index. The extra complication of solving the isomorphism problem for lattices is a dissuading factor, at least in implementing the use of this additional structure (we imagine that the time spent computing the subgroups would however still dominate the time solving the lattice isomorphism problem).

One is most interested in profinite rigidity in settings where the fundamental group determines the manifold. This is not true of knot complements in general, but it is true for complements of prime knots (modulo mirror symmetry). Thus Boileau and Friedl proposed [BF15] the question of whether the complements of prime knots are profinitely rigid. From an experimental point of view, however, this question is almost exactly the same as Question 5.1: of the 1 701 935 prime knots of crossing number at most 16 tabulated by Hoste, Thistlethwaite and Weeks [HTW98], only 32 are non-hyperbolic. We are in the process of applying our machinery to this collection of examples in any case. Fortunately, the knots are available in SnapPy with group presentations of quite low rank: 67% are rank 3 and 31% are rank 4, with only 10 examples of rank 6, the largest occurring. (For comparison, the standard Wirtinger presentation for a knot group given a knot diagram has as many generators as the diagram has crossings, which is prohibitive when equal to 16.)

Moreover, we are lucky that the unsigned Alexander polynomial (that is, the Alexander polynomial modulo multiplication by ± 1) has recently been shown to be a profinite invariant [Uek17]. This takes us most of the way: we computed the entropy of the unsigned Alexander polynomial amongst the prime knots of crossing number at most 16 to be 16.67, which is very close to the entropy of $\log_2(1701935) = 20.70$ that full discrimination of these knots would require. An interpretation of the gap of 4.03 is that a random knot lies in an unsigned Alexander polynomial equivalence class of size $2^{4.03} \approx 16$, where this average is computed as the geometric mean. Indeed, 49.6% of the knots are in an equivalence class of size at most 16. There are 140 261 knots, representing 8.2% of the total, that are each already distinguished from all the others by their unsigned Alexander polynomial, which leaves us with 1 561 674 knots.

Thus we are optimistic that by exploiting some knot theory we will be able to carry out this experiment to completion. It will, however, take much longer than the SnapPea census experiment. In particular, a random sample of 1000 of the 57 005 pairs of knots with the same Alexander polynomial (modulo sign) took approximately 4.5 hours of CPU time. At this rate, extrapolating generously – whereas we anticipate that the larger Alexander polynomial equivalence classes would take longer *per knot* – we have an estimated lower bound of 150 days of CPU time. Because the homology of cyclic covers will be periodic when all roots of the Alexander polynomial are roots of unity [Gor72], we cannot get away with using just these obvious subgroups, and will inevitably have to search for non-nilpotent finite quotients, and most likely finite simple quotients. Each such `GQuotients` search takes on the order of seconds; 1.5 million seconds is approximately 17 days.

Chapter 6

Minimal sizeable graphs

In this chapter we report briefly on the computational determination of minimal graphs with certain properties, called "sizeable graphs", arising in the study of finiteness properties of subgroups of hyperbolic groups.

Definition 6.1 (Sizeable graph, [Kro16])**.** A graph Γ is *sizeable* if it satisfies the following:

- Γ is bipartite on two sets A and B;
- Γ contains no cycles of length 4; *and*
- there exist partitions $A = A^+ \sqcup A^-$ and $B = B^+ \sqcup B^-$ such that the induced subgraph $\Gamma(A^s \sqcup B^t)$ is connected for all $s, t \in \{+, -\}$.

We call the 4 induced subgraphs $\Gamma(A^s \sqcup B^t)$ the *defining* subgraphs of Γ.

On the large scale, a sizeable graph "looks like" a 4-cycle: think of shrinking each A^s, B^t to a single vertex, and draw an edge to represent an induced subgraph that is connected. This runs contrary to containing no 4-cycle, which makes constructing such graphs difficult.

Suppose that each A^s and B^t has n edges. Connectivity of the defining subgraphs requires that they have average degree approximately 2 (so average degree 4 in the whole graph), since the sum of degrees over the $2n$ vertices must be at least $2(2n-1)$.

Proposition 6.2. *Let Γ be a random bipartite graph on $A \sqcup B$ with $|A| = |B| = 2n$, constructed by including any edge from A to B independently with probability $\frac{2}{n}$. Then the expected number of subgraphs of Γ isomorphic to the 4-cycle is $(1 - \frac{1}{2n})^2 64$.*

This underlines the difficulty of constructing such graphs.

Proof. The number of possible 4-cycles is $\binom{2n}{2}^2$, and the probability of any given 4-cycle occurring is $\left(\frac{2}{n}\right)^4$. $\square$

Motivation

Sizeable graphs are used to construct hyperbolic groups G such that $G \twoheadrightarrow \mathbb{Z}$ with kernel K which is finitely presented but not of type FP_3, so that K is non-hyperbolic [Lod17] [Kro16].

6.1 Theoretical bounds

Basic combinatorics

Proposition 6.3. *A sizeable graph with all 4 defining subgraphs cycles has at least 32 vertices.*

Proof. Consider a vertex $a^+ \in A^+$. It will have precisely 2 neighbours in each of B^+ and B^-. Let N denote the set of these 4 vertices. Each vertex $b \in N$ has precisely 2 neighbours in A^-. If some vertex $a^- \in A^-$ were a common neighbour of distinct vertices $b_0, b_1 \in N$, then this would give a 4-cycle (a^+, b_0, a^-, b_1). Thus there are 8 distinct vertices in A^- that are neighbours of vertices in N, so $|A^-| \geq 8$. Likewise $|A^+| \geq 8$ and similarly for $B^\pm$, so Γ has at least 32 vertices. $\square$

The Zarankiewicz Problem

We now relate the problem of determining the smallest sizeable graphs to the well-studied (but unsolved in general) *Zarankiewicz Problem*, using known results to get a lower bound essentially for free.

We first weaken the problem of finding sizeable graphs: rather than demand that the 4 defining subgraphs be connected, we require only that the whole graph contain at least $2(|A| + |B|) - 4$ edges. This is certainly a necessary condition to be sizeable, as a subgraph on n vertices can only be connected if it has $n - 1$ edges, so summing over the 4 defining subgraphs gives $2(|A| + |B|) - 4$.

Now instead of asking what is the minimum number of vertices we need to construct a bipartite graph containing no 4-cycle with sufficiently many edges, we can look at the problem the other way, and ask, for a given number of vertices, what is the maximum number of edges. This is an instance of the general Zarankiewicz problem of determining Zarankiewicz numbers.

Definition 6.4 ([DHS13, Definition 1.2]). A bipartite graph $G = (A, B; E)$ is called $K_{s,t}$-free if it does not contain s vertices in A and t vertices in B that span a subgraph isomorphic to the complete bipartite graph $K_{s,t}$. The maximum number of edges that a $K_{s,t}$-free bipartite graph of size (m, n) may have is the *Zarankiewicz number* $Z_{s,t}(m, n)$.

Even for $(s, t) = (2, 2)$ – the case we are interested in, corresponding to having no 4-cycles – not all Zarankiewicz numbers are known exactly, but they are known for the range of values relevant to us [DHS13, Table 1]. What matters is the range of m and n such that $Z_{2,2}(m, n) \geq 2(m + n - 2)$. This smallest value of $m + n$ for which this holds is 23: $Z_{2,2}(11, 12) = 42$ (and is the only possibility with $m + n = 23$ up to swapping m and n), whereas for $m + n = 22$ we have

(m, n)	$(7, 15)$	$(8, 14)$	$(9, 13)$	$(10, 12)$	$(11, 11)$
$Z_{2,2}(m, n)$	33	35	37	39	39

Corollary 6.5. *A sizeable graph has at least 23 vertices.*

The bound is already as good as the bound we achieved by applying the method of proof for Proposition 6.3 over all possible ways of partitioning the degrees in a tree (minimal connected graph), as opposed to the partition $(2, 2, \ldots, 2)$ that occurs in a cycle.

Remark 6.6. We tried several different encodings of the problem of finding minimal sizeable graphs (with no topological assumptions), which in theory could be solved using industrial software, but this was without success. A natural encoding is quadratic programming, but solving such a formulation is difficult. In particular, we phrased the problem as ILP (integer linear programming), SAT, and SET-COVER.

Remark 6.7. A natural class of bipartite graphs with no 4-cycles is the projective planes, where the bipartition of vertices is into points and lines from the projective geometry. There are no 4-cycles since this would give two points of intersection of a pair of lines. However, it is not possible to endow these vertices with partitions $A = A^+ \sqcup A^-$ and $B = B^+ \sqcup B^-$ – at least in the range that could improve on our result, namely order 2 and 3 giving graphs on 14 and 26 vertices respectively – such that the 4 defining subgraphs are connected, as verified by exhaustive search.

6.2 Computations and results

Previous examples of sizeable graphs were constructed by giving each defining subgraph the structure of a bipartite Cayley graph for a finite cyclic group. Phrased slightly differently, one labels vertices $\{a_0, \ldots a_{n-1}\}$ and $\{b_0, \ldots, b_{n-1}\}$ and then chooses integers d and joins each a_i to b_{i+d}, where the indices are computed modulo n. For the example of Lodha, $n = 11$ so the graph has 44 vertices [Lod17, Definition 8]. The example of Kropholler was a slight variation on this theme, with 37 vertices [Kro16].

By an exhaustive computational search, aided by the fact that one can fix the defining subgraphs $\Gamma(A^+ \sqcup B^+)$ and $\Gamma(A^- \sqcup B^-)$ without loss of generality, we determined the following:

Theorem 6.8. *The smallest sizeable graph with each of the 4 defining subgraphs a cycle has 36 vertices.*

This graph is built out of Cayley graphs, in the style of Lodha's example.

Remark 6.9. If the defining subgraphs are cycles then $|A^+| = |B^+| = |A^-| = |B^-|$, so in light of Proposition 6.3, the content of Theorem 6.8 is that the lower limit of 32 cannot be realized.

Theorem J. *The smallest sizeable graph with each of the 4 defining subgraphs a path has 31 vertices.*

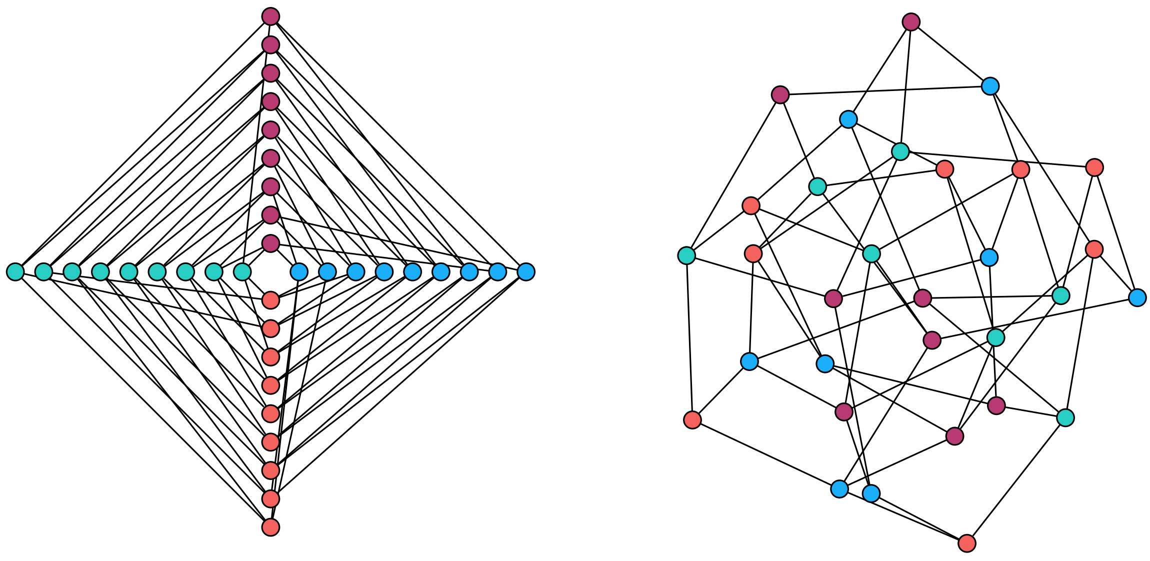


Figure 6.1: Minimal sizeable graphs

These graphs are indicated in Figure 6.1. As is apparent from the figure, the minimal cycle example can be arrange moreover to have the "Cayley graph" structure described above.

In ongoing work, we are attempting to find the minimal sizeable graphs without any assumptions on the topology of the defining subgraphs. We anticipate that they will be no smaller than our path example; intuitively, one expects that the "branching" of a tree which is not a path makes it easier for 4-cycles to arise.

Appendix A

The one-relator census

– On va voir?
– Non. C'est trop compliqué. Mais on exige de l'explorateur qu'il fournisse des preuves. S'il s'agit par example de la découverte d'une grosse montagne, on exige qu'il en rapporte de grosses pierres.

Antoine de Saint-Exupéry, *Le Petit Prince*

In this appendix we give a table showing the full one-relator census, which was the topic of Chapter 3. The columns list:

- the relator itself (the minimal representative, under the `shortlex` order, of its isomorphism class);
- a description of the group if known (which in most cases is a name for the group but also notes torsion groups which are not $\mathbb{Z} * \mathbb{Z}/n$);
- whether it has unbalanced Baumslag–Solitar subgroups;
- whether it is automatic; *and*
- whether it is either free-by-cyclic or an ascending HNN extension of a free group (determined via Brown's Criterion, see Remark 1.5).

To recall, the names include: BS for Baumslag–Solitar groups, T for torus knot complements, EBS for extended Baumslag–Solitar groups, DS for the Druţu–Sapir group, and $G_{(1,2)}$ for the Baumslag–Gersten group. Moreover, some 6 groups are identified as fundamental groups of cusped hyperbolic 3-manifolds (of small volume) in the SnapPea census. They include the figure eight knot complement `m004` and its sibling `m003` (of the same volume, which is minimal amongst cusped hyperbolic 3-manifolds). We also record some groups of interest arising elsewhere in the literature (see, for instance, Remark 3.7).

Relator	Description	BS?	Auto?	$F \rtimes \mathbb{Z}$?
1	F_2		y	
a	$\mathbb{Z}$		y	y
a^2	$\mathbb{Z} * \mathbb{Z}/2$		y	
a^3	$\mathbb{Z} * \mathbb{Z}/3$		y	
a^4	$\mathbb{Z} * \mathbb{Z}/4$		y	
a^2b^2	$\mathrm{BS}(1,-1)$, $T(2,2)$		y	y
$aba^{-1}b^{-1}$	$\mathrm{BS}(1,1)$		y	y
a^5	$\mathbb{Z} * \mathbb{Z}/5$		y	
a^3b^2	$T(2,3)$		y	y
$a^2ba^{-1}b^{-1}$	$\mathrm{BS}(1,2)$	y		asc
a^2bab^{-1}	$\mathrm{BS}(1,-2)$	y		asc
a^6	$\mathbb{Z} * \mathbb{Z}/6$		y	
a^4b^2	$T(2,4)$		y	y
$a^3ba^{-1}b^{-1}$	$\mathrm{BS}(1,3)$	y		asc
a^3bab^{-1}	$\mathrm{BS}(1,-3)$	y		asc
a^3b^3	$T(3,3)$		y	y
$a^2ba^{-2}b^{-1}$	$\mathrm{BS}(2,2)$		y	y
a^2bab^{-2}	$F_3 \rtimes \mathbb{Z}$ of [BC07]		y	y
$a^2ba^2b^{-1}$	$\mathrm{BS}(2,-2)$		y	
$a^2b^2a^{-1}b^{-1}$			y	y
a^7	$\mathbb{Z} * \mathbb{Z}/7$		y	
a^5b^2	$T(2,5)$		y	y
$a^4ba^{-1}b^{-1}$	$\mathrm{BS}(1,4)$	y		asc
a^4bab^{-1}	$\mathrm{BS}(1,-4)$	y		asc
a^4b^3	$T(3,4)$		y	y
$a^3ba^{-2}b^{-1}$	$\mathrm{BS}(2,3)$	y		
$a^3ba^{-1}b^{-2}$			y	y
$a^3ba^{-1}b^2$			y	y
a^3bab^{-2}			y	y
$a^3ba^2b^{-1}$	$\mathrm{BS}(2,-3)$	y		
$a^2ba^2b^{-2}$			y	y
$a^2b^2a^{-2}b^{-1}$	DS	y		asc
$a^2b^2ab^{-2}$	$\mathrm{EBS}(a,b^2,1,-2)$	y		asc
a^8	$\mathbb{Z} * \mathbb{Z}/8$		y	
a^6b^2	$T(2,6)$		y	y
$a^5ba^{-1}b^{-1}$	$\mathrm{BS}(1,5)$	y		asc
a^5bab^{-1}	$\mathrm{BS}(1,-5)$	y		asc
a^5b^3	$T(3,5)$		y	y
$a^4ba^{-2}b^{-1}$	$\mathrm{BS}(2,4)$	y		
$a^4ba^{-1}b^{-2}$			y	y
$a^4ba^{-1}b^2$			y	y
a^4bab^{-2}			y	y

$a^4ba^2b^{-1}$	$\mathrm{BS}(2,-4)$	y		
a^4b^4	$T(4,4)$		y	y
$a^3ba^{-3}b^{-1}$	$\mathrm{BS}(3,3)$		y	y
$a^3ba^{-2}b^{-2}$			y	y
$a^3ba^{-2}b^2$			y	y
a^3bab^{-3}			y	y
$a^3ba^2b^{-2}$			y	y
$a^3ba^3b^{-1}$	$\mathrm{BS}(3,-3)$		y	
a^3bab^3			y	asc
$a^3b^2a^{-1}b^{-2}$	$\mathrm{EBS}(a,b^2,1,3)$	y		asc
$a^3b^2a^{-1}b^2$			y	y
$a^3b^2ab^{-2}$	$\mathrm{EBS}(a,b^2,1,-3)$	y		asc
$a^3b^2ab^2$			y	y
$a^3b^3a^{-1}b^{-1}$			y	y
$a^2ba^{-1}b^{-2}a^{-1}b$	non-LERF [BKS87]		y	y
$a^2bab^{-1}a^{-1}b^2$			y	y
$a^2bab^{-1}ab^{-2}$			y	y
$a^2bab^{-1}ab^2$			y	asc
$a^2bab^2a^{-1}b^{-1}$			y	asc
$a^2bab^2ab^{-1}$			y	y
$a^2b^2a^{-2}b^{-2}$			y	y
$a^2b^2a^{-1}b^{-1}a^{-1}b^{-1}$			y	y
$a^2b^2ab^{-1}a^{-1}b$			y	
$a^2b^2a^2b^{-2}$			y	
$a^2b^2a^2b^2$	torsion		y	
$aba^{-1}b^{-1}aba^{-1}b^{-1}$	torsion		y	
a^9	$\mathbb{Z} * \mathbb{Z}/9$		y	
a^7b^2	$T(2,7)$		y	y
$a^6ba^{-1}b^{-1}$	$\mathrm{BS}(1,6)$	y		asc
a^6bab^{-1}	$\mathrm{BS}(1,-6)$	y		asc
a^6b^3	$T(3,6)$		y	y
$a^5ba^{-2}b^{-1}$	$\mathrm{BS}(2,5)$	y		
$a^5ba^{-1}b^{-2}$			y	y
$a^5ba^{-1}b^2$			y	y
a^5bab^{-2}			y	y
$a^5ba^2b^{-1}$	$\mathrm{BS}(2,-5)$	y		
a^5b^4	$T(4,5)$		y	y
$a^4ba^{-3}b^{-1}$	$\mathrm{BS}(3,4)$	y		
$a^4ba^{-2}b^{-2}$			y	y
$a^4ba^{-2}b^2$			y	y
$a^4ba^{-1}b^{-3}$			y	y
$a^4ba^{-1}b^3$			y	y
a^4bab^{-3}			y	y

$a^4ba^2b^{-2}$			y	y
$a^4ba^3b^{-1}$	$\mathrm{BS}(3,-4)$	y		
a^4bab^3			y	y
$a^4b^2a^{-1}b^{-2}$	$\mathrm{EBS}(a,b^2,1,4)$	y		asc
$a^4b^2a^{-1}b^2$			y	y
$a^4b^2ab^{-2}$	$\mathrm{EBS}(a,b^2,1,-4)$	y		asc
$a^4b^2ab^2$			y	y
$a^3ba^{-1}b^{-1}a^{-1}b^{-2}$			y	y
$a^3ba^{-1}b^{-2}a^{-1}b$	`m004(0,0)`		y	y
$a^3ba^{-1}b^{-1}ab^{-2}$			y	y
$a^3ba^{-1}bab^2$			y	y
$a^3ba^{-1}b^2ab$			y	y
$a^3bab^{-1}a^{-1}b^{-2}$			y	y
$a^3bab^{-1}a^{-1}b^2$			y	y
$a^3bab^{-2}a^{-1}b$	`m005(0,0)`		y	y
$a^3bab^{-2}ab$	`m003(0,0)`		y	y
$a^3bab^{-1}ab^{-2}$			y	y
$a^3bab^{-1}ab^2$			y	y
$a^3ba^2b^{-3}$			y	y
$a^3ba^3b^{-2}$			y	y
$a^3ba^2b^3$			y	y
$a^3baba^{-1}b^{-2}$			y	asc
$a^3baba^{-1}b^2$			y	y
a^3babab^{-2}			y	asc
$a^3b^2a^{-3}b^{-1}$	$\mathrm{EBS}(a,b^3,1,2)$	y		asc
$a^3b^2a^{-3}b$	$\mathrm{EBS}(a,b^3,1,-2)$	y		asc
$a^3b^2a^{-2}b^{-2}$	$\mathrm{EBS}(a,b^2,2,3)$	y		
$a^3b^2a^{-2}b^2$			y	y
$a^3b^2a^{-1}b^{-1}a^{-1}b^{-1}$			y	asc
$a^3b^2ab^{-1}a^{-1}b^{-1}$			y	asc
$a^3b^2ab^{-1}a^{-1}b$			y	asc
$a^3b^2ab^{-3}$			y	y
$a^3b^2a^2b^{-2}$	$\mathrm{EBS}(a,b^2,2,-3)$	y		
$a^3b^3a^{-2}b^{-1}$			y	y
$a^2ba^{-1}b^{-1}a^{-1}ba^{-1}b^{-1}$			y	
$a^2ba^{-1}b^{-1}a^{-1}bab^{-1}$	$G_{(1,2)}$	y		
$a^2ba^{-1}b^{-1}abab^{-1}$	$\mathrm{EBS}(a,bab^{-1},1,-2)$	y		
$a^2bab^{-1}a^{-1}bab^{-1}$			y	
$a^2bab^{-2}a^{-2}b$			y	asc
$a^2bab^{-2}a^{-1}b^2$			y	asc
$a^2bab^{-1}a^2b^{-2}$			y	y
$a^2ba^2b^{-1}a^{-1}b^{-2}$			y	y
$a^2ba^2ba^{-1}b^{-2}$			y	asc

$a^2ba^2bab^{-2}$		y	asc
$a^2baba^{-2}b^{-2}$	`m008(0,0)`	y	
$a^2baba^2b^{-2}$	`m006(0,0)`	y	
$a^2bab^2a^{-2}b^{-1}$		y	y
$a^2b^2a^2b^{-1}a^{-1}b^{-1}$	`m007(0,0)`	y	
$a^2b^2a^2b^{-1}a^{-1}b$		y	y

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